Rev. Téc. Ing. Univ. Zulia. Vol. 20, No. 2, 129-132, 1997

Some results involving the parabolic cylinder function

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Abstract

This paper is devoted to the study of the modified moments of the weight functions $x^{\mu} e^{-\frac{x^2}{4}} (\ln x)^p$, p=1,2, on [0,∞), with respect to the parabolic cylinder function $D_\nu(x)$. A procedure for a numerical computation is discussed. Special cases are mentioned.

Key words: Integrals, parabolic cylinder function, modified moments.

Algunos resultados que involucran la función cilíndrica parabólica

Resumen

En este trabajo, se estudian los momentos modificados de las funciones de peso $x^{\mu} e^{-\frac{x^2}{4}} (\ln x)^p$, p=1,2, sobre [0,∞), con respecto a la función cilíndrica parabólica $D_\nu(x)$. Un procedimiento de cálculo y varios casos particulares son presentados.

Palabras claves: Integrales, función cilíndrica parabólica, momentos modificados.

Introduction

Several authors have studied and solved in a series of papers the evaluation of integrals involving Special Functions. Many of them obtained algorithms concerning integrals of orthogonal polynomials [2,4,5,8,9,10,11,13], Jacobi functions [7] or hypergeometric functions [1,14,15].

More recently Gonzales and Kalla [6] and Prieto and Galué [15] have evaluated the moments of the weight functions $x^{\lambda} e^{-px} (\ln x)^t$, p > 0, t = 1,2 on [0,∞) with respect to the product of generalized Laguerre functions and hypergeometric functions respectively.

Our object in the present note is to compute the integrals

$$I_\nu^{(p)}(\mu) = \int_0^\infty x^{\mu} e^{-x^2/4} (\ln x)^p D_\nu(x)dx \tag{1}$$

$\mu > -1$, $p = 1,2.$

Results of Gatteschi [4] are special cases of integral formulae of this paper.

Since (1) follows by differentiating with respect to the parameter μ the integral

$$J_\nu(\mu) = \int_0^\infty x^{\mu} e^{-x^2/4} D_\nu(x)dx , \tag{2}$$

that is

$$I_\nu^{(p)}(\mu) = \frac{\partial^p}{\partial \mu^p} J_\nu(\mu), \; p = 1,2... , \tag{3}$$

we will evaluate the sequence of integrals (2) and (3) obtained by changing ν into $\nu + k$, where $k = 1,2,\ldots$.

Integral $J_\nu(\mu)$ and related derivatives

Using the following formula

$$\int_0^\infty x^{\alpha-1} e^{-c^2x^2/4} D_\nu(cx)dx = \frac{2^{(\nu-\alpha)/2}\sqrt{\pi}}{c^\alpha}\frac{\Gamma(\alpha)}{\Gamma(\frac{1+\alpha-\nu}{2})}, \quad (4)$$

$Re(\alpha) > 0$, $|\arg c| < \frac{\pi}{4}$.

(see 2.11.3.11 of [16]), the moment (2) reduces to

$$J_\nu(\mu) = \sqrt{\pi}\, 2^{(\nu-\mu-1)/2}\frac{\Gamma(1+\mu)}{\Gamma(1+\frac{\mu-\nu}{2})} \quad (5)$$

To evaluate the integrals (1), we compute the first and the second derivatives of (5) with respect to μ. We have

$$I_\nu^{(1)}(\mu) = \frac{\partial}{\partial\mu}J_\nu(\mu) = J_\nu(\mu)[-\frac{1}{2}\ln 2 + \psi(1+\mu) - \frac{1}{2}\psi(1-\frac{\nu}{2}+\frac{\mu}{2})] \quad (6)$$

$$I_\nu^{(2)}(\mu) = \frac{\partial^2}{\partial\mu^2}J_\nu(\mu) = J_\nu(\mu)\{[-\frac{1}{2}\ln 2 + \psi(1+\mu) - \frac{1}{2}\psi(1-\frac{\nu}{2}+\frac{\mu}{2})]^2 + \psi'(1+\mu) - \frac{1}{4}\psi'(1-\frac{\nu}{2}+\frac{\mu}{2})\}. \quad (7)$$

Recurrence relations

Now we illustrate some recursive formulas to evaluate the moments $J_{\nu+m}(\mu)$ and $I_{\nu+m}^{(p)}(\mu)$, $p = 1,2$, with m integer positive number.

From the recurrence formula of the Gamma function, it can be shown that

$$J_{\nu+m}(\mu) = 2^n(-1)^n\left(\frac{\nu'-\mu}{2}\right)_n J_{\nu'}(\mu)\ , \quad (8)$$

where $n=\left[\frac{m}{2}\right]$ and

$$\nu' = \begin{cases}\nu\ , & \text{if } m=2n\\ \nu+1\ , & \text{if } m=2n+1.\end{cases} \quad (9)$$

Consequently, by putting $\alpha_0^{(\nu')} = J_{\nu'}(\mu)$ and $\alpha_n^{(\nu')} = J_{\nu+m}(\mu)$, it is immediate to verify the simple relation

$$\alpha_{n+1}^{(\nu')} = -2\left(\frac{\nu'-\mu}{2}+n\right)\alpha_n^{(\nu')}\ ,\quad n = 0,1,\ldots\ . \quad (10)$$

Then, for the computation of (6) when ν is substituted by $\nu + m$, with the aid of the Psi-function recursive formula

$$\psi(z+1) = \psi(z) + \frac{1}{z}, \quad (11)$$

we may construct the following sequences

$$\beta_{n+1}^{(\nu')} = \beta_n^{(\nu')} - \frac{1}{2(\frac{\nu'-\mu}{2}+n)},\quad n = 0,1,\ldots, \quad (12)$$

$$\beta_0^{(\nu')} = -\frac{1}{2}\ln 2 + \psi(1+\mu) - \frac{1}{2}\psi(1-\frac{\nu'-\mu}{2}), \quad (13)$$

with ν' given by (9). Hence, taking into account (10), we have

$$I_{\nu+m}^{(1)}(\mu) = \alpha_n^{(\nu')}\beta_n^{(\nu')}\ ,\quad n = [\tfrac{m}{2}]. \quad (14)$$

Similarly, by replacing in (7) ν with $\nu + m$ and remembering that

$$\psi'(1+z) = \psi'(z) - \frac{1}{z^2}, \quad (15)$$

together with (9), (10) and (12), we obtain the recurrence relationships

$$\gamma_{n+1}^{(\nu')} = \gamma_n^{(\nu')} - \frac{1}{4(\frac{\nu'-\mu}{2}+n)^2}, \quad (16)$$

$$\gamma_0^{(\nu')} = \psi'(1+\mu) - \frac{1}{4}\psi'(1-\frac{\nu'-\mu}{2}), \quad (17)$$

and finally the algorithm

$$I^{(2)}_{\nu+m}(\mu)=\alpha_n^{(\nu')}\{\beta_n^{(\nu')2}+\gamma_n^{(\nu')}\},\ n=[\tfrac{m}{2}]\ . \qquad (18)$$

Particular cases

We mention here some special cases of the results established in the previous sections.

a) $\nu = k$.

It is well known that $D_\nu(x)$ reduces to Laguerre polynomials when ν is an integer number. More precisely

$$D_{2m}(x)=\frac{\sqrt{\pi}2^m e^{-\frac{x^2}{4}}}{\Gamma(\frac{1}{2}-m)\binom{m-\frac{1}{2}}{m}}L_m^{(-\frac{1}{2})}(\tfrac{x^2}{2}), \qquad (19)$$

$$D_{2m+1}(x)=-\frac{\sqrt{\pi}2^{m+1}xe^{-\frac{x^2}{4}}}{\Gamma(-\frac{1}{2}-m)\binom{m+\frac{1}{2}}{m}}L_m^{(\frac{1}{2})}(\tfrac{x^2}{2}). \qquad (20)$$

Then, if $\nu = 2m$, by using (5) we have

$$J_{2m}(\nu)=\frac{\sqrt{\pi}2^{m+\frac{\mu-1}{2}}}{\Gamma(\frac{1}{2}-m)\binom{m-\frac{1}{2}}{m}}\int_0^\infty t^{\frac{\mu-1}{2}}e^{-t}L_m^{(-\frac{1}{2})}(t)\,dt=$$

$$=\sqrt{\pi}2^{m-\frac{\mu+1}{2}}\frac{\Gamma(1+\mu)}{\Gamma(1-m+\frac{\mu}{2})} \qquad (21)$$

and consequently

$$\int_0^\infty t^{\frac{\mu-1}{2}}e^{-t}L_m^{(-\frac{1}{2})}(t)\,dt=\frac{(-1)^m\Gamma(\frac{\mu}{2}+1)\Gamma(\frac{\mu+1}{2})}{m!\ \Gamma(\frac{\mu}{2}-m+1)}. \qquad (22)$$

This result may be deduced from a Gatteschi formula [4].

Analogously, if $\nu = 2m + 1$ we get

$$\int_0^\infty t^{\frac{\mu}{2}}e^{-t}L_m^{(\frac{1}{2})}(t)\,dt=\frac{(-1)^m\Gamma(\frac{\mu}{2}+1)\Gamma(\frac{\mu+1}{2})}{m!\ \Gamma(\frac{\mu}{2}-m+\frac{1}{2})}. \qquad (23)$$

b) $\nu - \mu = 2k,\ k \geq 1$.

Since the function $\Gamma(x)$ and $\psi(x)$ have singularities when $x = 0,-1,-2,\ldots$, results (6) and (7) are valid provided that $\nu - \mu$ is not a positive even number.

In this case, to evaluate the moments (1) we take the limit and use the formula

$$\lim_{\varepsilon\to 0}\frac{\psi(-r+\varepsilon)}{\Gamma(-r+\varepsilon)}=(-1)^{r-1}r!\ , \qquad (24)$$

where r is an integer positive number.

So, we obtain

$$I^{(1)}_{\mu+2k}(\mu)=(-1)^{k+1}\sqrt{\pi}2^{k-\frac{3}{2}}\Gamma(1+\mu)\Gamma(k). \qquad (25)$$

To evaluate the second derivative (7), we can write

$$I^{(2)}_{\mu+2k}(\mu)=\sqrt{\pi}2^{k-\frac{1}{2}}\Gamma(1+\mu)\left[\tfrac{1}{4}A(\mu)+B(\mu)\right], \qquad (26)$$

where

$$A(\mu)=\lim_{\varepsilon\to 0}\frac{\psi^2(1-k+\varepsilon)-\psi'(1-k+\varepsilon)}{\Gamma(1-k+\varepsilon)}, \qquad (27)$$

and

$$B(\mu)=\left[\frac{1}{2}\ln 2-\psi(1+\mu)\right]\lim_{\varepsilon\to 0}\frac{\psi(1-k+\varepsilon)}{\Gamma(1-k+\varepsilon)}. \qquad (28)$$

Hence

$$I^{(2)}_{\mu+2k}(\mu)=(-1)^k\sqrt{\pi}2^{k-\frac{3}{2}}\Gamma(1+\mu)\Gamma(k).$$
$$\{\psi(k)+\ln 2-2\psi(1+\mu)\}. \qquad (29)$$

We note that, to obtain

$$A(\mu)=(-1)^k 2\,\psi(k)\Gamma(k), \qquad (30)$$

we have used the following series expansions

$$\Gamma(x)=\frac{(-1)^r}{r!}\frac{1}{x+r}+\sum_{k=0}^\infty a_k(x+r)^k,$$

$$\psi(x)=-\frac{1}{x+r}+\psi(1+r)+\sum_{k=1}^\infty b_k(x+r)^k, \qquad (31)$$

valid for $|x+r|<1$.

Acknowledgement

Work supported by the Consiglio Nazionale delle Ricerche of Italy and by the Ministero dell'Universitá e della Ricerca Scientifica e Tecnologica of Italy.

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Recibido el 18 de Octubre de 1995
En forma revisada el 19 de Mayo de 1997