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A note on some integrals involving two associated Laguerre polynomials

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Abstract

A closed form expression in terms of Appells function F_2 is derived for the integral of two Associated Laguerre polynomials of the form

$$\int_0^\infty e^{-sx} x^{\lambda-1} L_n^a(\alpha x) L_m^b(\beta x) dx$$

Some integral formulas corresponding to special values of the parameters and the arguments are deduced, the results are believed to be new.

Key words: Laguerre polynomials, Appell's functions, hypergeometric functions.

Una nota sobre algunos integrales que involucran dos polinomios asociados de Laguerre

Resumen

Se deriva una expresión en forma cerrada en términos de la función F_2 de Appell para la integral de dos polinomios asociados de Laguerre de la forma

$$\int_0^\infty e^{-sx} x^{\lambda-1} L_n^a(\alpha x) L_m^b(\beta x) dx$$

Se deducen algunas fórmulas integrales correspondientes a valores especiales de los parámetros y del argumento. Se cree que los resultados son nuevos.

Palabras claves: Polinomios Laguerre, funciones Appell, funciones hipergeométricas.

Introduction

This paper is concerned with the evaluation of the Laplace type integral

$$I = \int_0^\infty e^{-sx} x^{\lambda-1} L_n^a(\alpha x) L_m^b(\beta x) dx , \qquad (1)$$

where L_n^a is the associated Laguerre polynomial. For specific values of the parameters, the integral (1) encountered in various mathematical physics problems, these includes, the N-dimensional quantum hydrogen atom, and harmonic oscillator [1-2], the electron gas in magnetic field [3], and others [4-6]. Some of these integrals are tabulated [7-8].

As far as we know, the most general integral of the type (1), which has been evaluated in closed form and asymptotically [3] is when $s = \alpha = \beta = 1$, and this integral is re-evaluated recently [9].

In section 2 the integral (1) is evaluated in closed form in terms of Appell's function F_2, and in section 3 and through the use of reduction formulas of F_2 for special values of the parameters and arguments, some integral formulas are derived.

Evaluation of the Integral I

The associated Laguerre polynomial is defined as

$$L_m^b(z) = \frac{(b+1)_m}{m!}\ {}_1F_1[-m; b+1, z]\ ,\ \text{Re}\, b > -1 \quad (2)$$

wherem ${}_1F_1$ is the confluent hypergeometric function which is a special case of the generalized hypergeometric function

$$ {}_pF_q[a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z] = \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j \ldots (a_p)_j}{j!(b_1)_j (b_2)_j \ldots (b_p)_j} z^j \quad (3)$$

where

$$(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)} = (a)(a+1)\ldots(a+j-1),\ (a)_0 = 1. \quad (4)$$

The infinite series in (3) terminates if one of the a_j's is zero or negative integer. In this work we consider m and n to be non-negative integers and all other parameters to be complex. Thus for the integral (1) to converge we assume that

$$\text{Re}\ s > 0\ ,\ \text{Re}\lambda > 0,\ \text{Re}\ a > -1,\ \text{Re}\ b > -1\ . \quad (5)$$

Substituting

$$L_m^b(\beta x) = \frac{(b+1)_m}{m!} \sum_{k=0}^{m} \frac{(-m)_k}{k!(b+1)_k} \beta^k x^k, \quad (6)$$

in (1) and interchanging the sum and integral we get

$$I = \frac{(b+1)_m}{m!} \sum_{k=0}^{m} \frac{(-m)_k \beta^k}{k!(b+1)_k} \int_0^{\infty} e^{-sx} L_n^a(\alpha x) x^{\lambda+k-1} dx, \quad (7)$$

which upon using [7,p.844] becomes

$$I = \frac{(a+1)_n (b+1)_m}{n! m! s^{\lambda}} \cdot \sum_{k=0}^{m} \frac{(-m)_k \Gamma(\lambda+k)}{k!(b+1)_k} \left(\frac{\beta}{s}\right)^k {}_2F_1\left[-n, \lambda+k; a+1; \frac{\alpha}{s}\right]\ . \quad (8)$$

Expanding the functions ${}_2F_1$ using (3) and making use of the identity

$$(\lambda+k)_j = \frac{\Gamma(\lambda)(\lambda)_{k+j}}{\Gamma(\lambda+k)} \quad (9)$$

we obtain

$$I = \frac{(a+1)_n (b+1)_m \Gamma(\lambda)}{n! m! s^{\lambda}} \cdot \sum_{k=0}^{m} \sum_{j=0}^{n} \frac{(-m)_k (-n)_j (\lambda)_{k+j}}{k! j! (b+1)_k (a+1)_j} \left(\frac{\beta}{s}\right)^k \left(\frac{\alpha}{s}\right)^j\ . \quad (10)$$

The double series in (10) is Appell's function F_2 defined as [10]

$$F_2[A, B, B'; C, C'; x, y] = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(A)_{k+j} (B)_k (B')_j}{k! j! (C)_k (C')_j} x^k y^j \quad (11)$$

which converges for $|x| + |y| < 1$. Thus, we obtain the main formula

$$I = \frac{(a+1)_n (b+1)_m \Gamma(\lambda)}{n! m! s^{\lambda}} F_2\left[\lambda, -n, -m; a+1, b+1; \frac{\alpha}{s}, \frac{\beta}{s}\right]\ . \quad (12)$$

Special Cases

Appell's function F_2 reduces to simpler functions for special values of its parameters and arguments. In the following we consider two special cases which lead to new integral formulas.

Case $s = \alpha$

When one of the arguments of Appell's function F_2 is unity it reduces to ${}_3F_2$ function, in particular, it is easy to show that

$$F_2[A; B, B'; C, C'; 1, y] = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} \cdot {}_3F_2\left[\begin{matrix} A, B', 1+A-C \\ C', 1+A+B-C \end{matrix}; y\right] \quad (13)$$

hence, when $s=\alpha$ in (12) we obtain

$$\int_0^{\infty} e^{-\alpha x} x^{\lambda-1} L_n^a(\alpha x) L_m^b(\beta x) dx = \frac{(b+1)_m (a+1-\lambda)_n \Gamma(\lambda)}{n! m! \alpha^{\lambda}} \cdot {}_3F_2\left[\begin{matrix} \lambda, -m, \lambda-a \\ b+1, \lambda-a-n \end{matrix}; \frac{\beta}{\alpha}\right] \quad (14)$$

provided that $|\beta| < |\alpha|$ and $\text{Re}(\alpha) > 0$.

Case $s = \alpha + \beta$

One of the transformation of the F_2 functions is [10]

$$F_2[A; B, B'; C, C'; x, y] = (1-y)^{-A} F_2\left[A; B; C'-B'; C, C'; \frac{x}{1-y}, \frac{y}{y-1}\right] \quad (15)$$

where $|x| + |y| < |1-y|$, which gives the alternate formula

$$I = \frac{(a+1)_n (b+1)_m \Gamma(\lambda)}{n!\ m!\ (s-\beta)^{\lambda}} \cdot F_2\left[\lambda; -n, m+b+1; a+1, b+1; \frac{\alpha}{s-\beta}, \frac{\beta}{\beta-s}\right]\ . \quad (16)$$

Substituting $s=\alpha+\beta$ in (16) and using (13) we obtain

$$\int_0^\infty e^{-(\alpha+\beta)x} x^{\lambda-1} L_n^a(\alpha x) L_m^b(\beta x) dx =$$

$$\frac{(b+1)_m (a-\lambda+1)_n \Gamma(\lambda)}{n! \; m! \alpha^\lambda} \; {}_3F_2\left[\begin{matrix}\lambda, b+m+1, \lambda-a \\ b+1, \lambda-a-n\end{matrix}; -\frac{\beta}{\alpha}\right] \quad (17)$$

provided that $|\beta| < |\alpha|$ and $\mathrm{Re}(\alpha+\beta) > 0$.

Using the symmetry of the integral, analogue formulae for (14) and (17) can be derived for the case $|\beta| > |\alpha|$ by interchanging $a \longleftrightarrow b$, $m \longleftrightarrow n$ and $\alpha \longleftrightarrow \beta$. Furthermore, several integral formulae can be deduced as special cases of (14) and (17) using the reduction formulas of ${}_3F_2$ function.

Conclusion

We have derived a closed form expression for the integral (1) from which we have deduced two integral formulas for special values of the parameters, these formulas are believed to be new.

References

1. Nieto M.M.: Hydrogen atom and relativistic pi-mesic atom in N-Space dimensions. Am.J.Phys., Vol. 47, No. 12, (1979), 1076-1072.
2. Nieto M.M. and Simmons L.M.: Eigenstates, coherent states, and uncertainty products for the Morse oscillator. Phys. Rev. A, Vol. 19, No. 2, (1979), 438-444.
3. Glasser M.L. and Horing N.J.M.: Ground-state energy of a two-dimensional electron gas in a magnetic field: Hartree-Fock approximation. Phys. Rev. B, Vol. 31, No. 7, (1985), 4603-4611.
4. Ismail M.E.H., Letessier J., and Valent G.: Linear birth and death models and associated Laguerre polynomials. J. Approx. Theory, Vol. 55, (1988).
5. Askey R., and Gasper G.: Certain rational functions whose power series have positive coefficients. Amer. Math. Monthly, Vol.79, (1972), 327-341.
6. Kugerl G.: Laguerre polynomial solution of the non-linear Boltzmann equation for the hard-sphere model. J. Appl. Phys (ZAMP), Vol. 40 (1989), 816-827.
7. Gradshteyn I.S., Ryzhik I.M: Tables of Integrals, Series, and Products. Academic press, New York, 1965.
8. Erdelyi A., Magnus W., Oberhettinger F., and Tricomi F.G.: Tables of Integral Transforms, Vol. II, Mc Graw-Hill, New York, 1954.
9. Mavromatis H.A.: An interesting new result involving Associated Laguerre polynomials. Inter. J. Computer Math., Vol. 36, (1990), 257-261.
10. Appell P., and Kampe' de Fe'riet J.: Fonction Hypergeometriques et Hyperspheriques, Gauthier-Villars, Paris, 1926.

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