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Some integral operators of Buschman-Erdélyi type

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Abstract

New integral operators involving the generalized Legendre function $P_k^{m,n}(z)$ are introduced. Some integrals are evaluated.

Key words: Operators, integrals, Legendre function.

Algunos operadores integrales de tipo Buschman-Erdélyi

Resumen

Se introducen nuevos operadores integrales que involucran la función generalizada de Legendre $P_k^{m,n}(z)$. Se evalúan algunas integrales.

Palabras claves: Operadores, integrales, función de Legendre.

Introduction

The classical integral transforms can be divided in two classes: the transforms of convolution type and the index integral transforms with special functions as the kernels [1-9].

The integral transforms with the Meijer's G- function or the Fox's H- function as the kernel [9, 10-16] are of great interest due to the general character of the kernel. A study of these transforms provide us deeper understanding of special integral transforms with simple kernels (which can be represented in terms of H- function).

Buschman-Erdelyi integral operators have the following form [17-20]:

$$Bf(x) = \int_0^x (x^2 - t^2)^{-\mu/2} \, P_\nu^\mu\left(\frac{x}{t}\right) f(t)\, dt \tag{1}$$

$$Ef(x) = \int_0^x (x^2 - t^2)^{-\mu/2} \, P_\nu^\mu\left(\frac{t}{x}\right) f(t)\, dt \tag{2}$$

where $P_\nu^\mu(z)$ is the Legendre function of the first kind, f(x) is the locally summable function, it satisfies to necessary conditions as $x \to 0$ and, $x \to \infty$; μ, ν are complex numbers, $\operatorname{Re}\mu < 1$ and, $\operatorname{Re}\nu \geq -\frac{1}{2}$.

Let us notice that these operators are also known with the integral limits from x to ∞ [17]-[20].

Interesting special cases of (1), (2) are studied in [21]-[26].

The operators of this type are important in mathematical physics (Dirichlet's problem for the Euler-Poisson-Darboux equation in the quadrant-plane) [27], in the theory of Radon's transform [28], [29], in the theory of the elliptic equations with singural points [30]-[33], etc.

The generalized associated functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ are two linear independent solutions of the following differential equation [34]:

$$(1-z^2)\,u''-2zu'+\left(k(k+1)-\frac{m^2}{2(1-z)}-\frac{n^2}{2(1+z)}\right)u=0 \quad (3)$$

where k,m,n, are complex parameters. These solutions give rise to the definition of a class of functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ which for m=n are the same as the well-known associated Legendre functions $P_k^m(z)$ and $Q_k^m(z)$ respectively.

Functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ belong to the class of hypergeometric functions. Some integral representations for $P_k^{m,n}(z)$, $Q_k^{m,n}(z)$ are established in [35] - [37].

The functions $P_k^{m,n}(z)$, $Q_k^{m,n}(z)$ arise in the solution of a sufficiently wide class of boundary value problems of mathematical physics such as, mechanics of the continuous medium; in more complicated system of the othogonal coordinates (the generate ellipsoidal, torodial, bipolar, spherical etc).

Integral Operators

We shall consider some integral operators with generalized associated Legendre's functions $P_k^{m,n}(z)$.

Let us introduce the following integral operator:

$$Pf(x)=\int_0^{\infty}(t+x)^{n/2}\,(t-x)^{m/2}\,P_k^{m,n}\left(\frac{t}{x}\right)f(t)\,dt \quad (4)$$

where Re $m<\frac{1}{2}, \frac{3}{2}m-\frac{n}{2}-1<k<\frac{-(m+n)}{2}$, $x>0$

and $f\in L_p(0,\infty)$, $1<p<\infty$, $P_k^{m,n}(z)$ is the generalized associated Legendre function.

Theorem 1

If $k+\frac{m+n}{2}>-1$, $m<n<m+1$, $\frac{3}{2}m-\frac{n}{2}-1<k<\frac{-(m+n)}{2}$, Re $m<\frac{1}{2}$, then the kernel of the integral operator (4) has the following integral representation:

$$(t+x)^{n/2}\,(t-x)^{m/2}\,P_k^{m,n}\left(\frac{t}{x}\right)Cx^{(n-m+1)/2}\,*$$

$$*\int_0^{\infty}e^{-t\xi}\xi^{-m-1/2}K_\nu(x\xi)\,{}_1F_1\!\left(n-m;k-\frac{3}{2}m+\frac{n}{2}+1;(t-x)\xi\right)d\xi \quad (5)$$

where

$$C=\sqrt{\frac{2}{\pi}}\,2^{n-m}\,\Gamma^{-1}\!\left(-k-\frac{m+n}{2}\right)\Gamma^{-1}\!\left(k-\frac{3}{2}m+\frac{n}{2}+1\right), \quad (6)$$

$\nu=k-(m-n-1)/2$

K_ν is the modified Bessel function, ${}_1F_1(a;c;z)=\Phi(a;c;z)$ is the confluent hypergeometric function.

Proof. Using the integral representation for $K_\nu(xt)$ [38]:

$$K_\nu(xt)=\sqrt{\pi}\,\Gamma^{-1}\!\left(\nu+\frac{1}{2}\right)t^{\nu}(2x)^{-\nu}\int_x^{\infty}e^{-t\xi}(\xi^2-x^2)^{\nu-1}d\xi, \quad (7)$$

and the application of the Laplace transform to $K_\nu(xt)$ [39], leads to:

$$P_\nu^{-\mu}\left(\frac{t}{x}\right)=\sqrt{\frac{2x}{\pi}}\,\frac{(t^2-x^2)^{\mu/2}}{\Gamma(\mu-\nu)\Gamma(\mu+\nu+1)}\int_x^{\infty}e^{-t\xi}\xi^{\mu-1}K_{\nu+1/2}(x\xi)d\xi \quad (8)$$

$Re(\nu)>\frac{-1}{2}$, $Re(\mu-\nu)>0$, $Re(\mu+\nu)>-1$, $Re(t)>0$.

According to the result [34], the function $P_k^{m,n}(z)$ can be represented in the form:

$$P_k^{m,n}(z)=\frac{2^{n-1}(z+1)^{(m-n)/2}(z^2-1)^{m/2}}{i\sqrt{\pi}\left(\frac{1}{2}-m\right)\cos(\pi m)}\int_0^{(a+\frac{1}{a}-)}\omega^{k+(m+n)/2}(1-2z\omega+\omega^2)^{-m-1/2}\,*$$

$$*\,{}_2F_1\!\left(-k-\frac{m+n}{2},n-m;\frac{1}{2}-m;\frac{-(\omega^2-2z\omega+1)}{2\omega(z+1)}\right)d\omega \quad (9)$$

where the integral is written with Pochhammer notations.

If $a=z+\sqrt{z^2-1}$ and the contour of integration is such that $|arg(\omega)|<\pi$, then $P_k^{m,n}(z)$ can be expressed in terms of the usual Legendre function $P_k^m(z)$ [34].

By virture of (8), (9), we get:

$$P_k^{m,n}\left(\frac{t}{x}\right)=\sqrt{\frac{2x}{\pi}}\,2^{n-m}\left(\frac{t+x}{x}\right)^2 \cdot$$
$$(t^2-x^2)^2\,\Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma(m-n+1)\cdot$$
$$\int_0^\infty e^{-t\xi}\xi^{-m-1/2}K_{k-(m-n-1)/2}(x\xi)\left[\sum_{i=0}^{\infty}\left[\left(t^2-x^2\right)^{1/2}\xi^i\left(\frac{t-x}{t+x}\right)^{1/2}\right]\cdot\right.$$
$$\left.\left[\frac{\Gamma^{-1}\left(i-k-\frac{m+n}{2}\right)\Gamma^{-1}\left(i+k-\frac{3}{2}m+\frac{n}{2}+1\right)}{\Gamma\left(k+\frac{m+n}{2}-i+1\right)}\right]\right]d\xi \quad (10)$$

Using [40],

$$\frac{\Gamma(a)}{\Gamma(a-i)}=(-1)^i\frac{\Gamma(i-a+1)}{\Gamma(1-a)},\quad (a)_i=\frac{\Gamma(a+i)}{\Gamma(a)}$$

We obtain,

$$P_k^{m,n}\left(\frac{t}{x}\right)=\sqrt{\frac{2}{\pi}}\,x^{(1-m+n)/2}\,2^{n-m}\,(t+x)^{-n/2}(t-x)^{-m/2}.$$
$$\Gamma^{-1}\left(-k-\frac{m+n}{2}\right)\Gamma^{-1}\left(k-\frac{3}{2}m+\frac{n}{2}+1\right).$$
$$\int_0^\infty e^{-t\xi}\xi^{-m-1/2}K_{k-(m-n)/2+1/2}(x\xi).$$
$$\Phi\left(n-m;k-\frac{3}{2}m+\frac{n}{2}+1;(t-x)\xi\right)d\xi\,. \quad (11)$$

Corollary. If Re $m<\frac{1}{2}$, $\frac{3}{2}m-\frac{n}{2}-1<k<-\frac{(m+n)}{2}$, $x>0$, $f\in L_p(0,\infty)$, $1<p<\infty$, then the integral operation (4) belongs to $L_p(0,\infty)$.

Further we introduce the following integral operator

$$\tilde{P}f(x)=\int_0^x (x-t)^{-m/2}(x+t)^{-n/2}\,P_k^{m,n}\left(\frac{x}{t}\right)f(t)\,dt \quad (12)$$

where $m<1$, $n<1$, $\frac{m-n}{2}-1<k<-\frac{(m+n)}{2}$.

Theorem 2

If $\alpha>0$, $x\in[a;b]$, $m<1$, $n<1$, $0<t<x$, $\frac{m-n}{2}-1<k\frac{-(m+n)}{2}$, then the kernel of the integral operator (12) has the following integral representation:

$$(x-t)^{-m/2}(x+t)^{-n/2}P_k^{m,n}\left(\frac{x}{t}\right)H(x-t)=2^{-k-(m-n)/2}t^{-k}.$$
$$I_x^{k-\frac{m+n}{2}}\left\{\frac{H(x-t)}{\Gamma\left(k-\frac{m-n}{2}+1\right)}(x^2-t^2)^{k+(m-n)/2}(x-t)^{n-m}\right\}, \quad (13)$$

where I_x^α is the Riemann-Liouville operator of fractional integration [9],

$$I_x^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt,$$

H(x) is a Heviside unit function.

Proof. We have [34];

$$P_k^{m,n}(z)=\frac{2^{-k-(m-n)/2}(z+1)^{k+m/2}(z-1)^{-m/2}}{\Gamma(1-m)}$$
$${}_2F_1\left(-k-\frac{m-n}{2};-k-\frac{m+n}{2};1-m;\frac{z-1}{z+1}\right),(m<1)\quad(14)$$

Further using the formula from [40] we rewrite (14) as,

$$P_k^{m,n}(z)=\frac{(z-1)^{-m/2}(z+1)^{n/2}}{2^{k+(m-n)/2}\,\Gamma\left(-k-\frac{m+n}{2}\right)\Gamma\left(k-\frac{m-n}{2}+1\right)}$$
$$\int_0^1 \xi^{-k-(m+n)/2-1}(1-\xi)^{k-(m-n)/2}\left((z+1)-(z-1)\xi\right)^{k+(m-n)/2}d\xi,\quad(15)$$
$$\left(\frac{m-n}{2}-1<k<-\frac{m+n}{2}\right)$$

Here the integral is considered as the principal value.

Clearly that the integral from (15) is equal to

$$\int_0^1 \xi^{-k-(m+n)/2-1}(1-\xi)^{k-(m-n)/2}\left((x+t)-(x-t)\xi\right)^{k+(m-n)/2}d\xi=\quad(16)$$
$$2^{k+(m-n)/2}\Gamma\left(-k-\frac{m+n}{2}\right)\Gamma\left(k-\frac{m-n}{2}+1\right)\frac{(x-t)^{m/2}}{(x+t)^{n/2}}t^k\,P_k^{m,n}\left(\frac{x}{t}\right).$$

Let $a\le t<x\le b$ and consider the following expression:

$$I_x^{\alpha}\left\{\frac{H(x-t)}{\Gamma\left(k-\frac{m-n}{2}+1\right)}(x^2-t^2)^{k+(m-n)/2}(x-t)^{n-m}\right\}, \tag{17}$$

where I_x^{α} is identified above. We see that

$$I_x^{\alpha}\left\{\frac{H(x-t)}{\Gamma\left(k-\frac{m-n}{2}+1\right)}(x^2-t^2)^{k+(m-n)/2}(x-t)^{n-m}\right\}=$$

$$=\frac{\Gamma^{-1}(\alpha)}{\Gamma\left(k-\frac{m-n}{2}+1\right)}\int_t^x(x-\xi)^{\alpha-1}(\xi^2-t^2)^{k+(m-n)/2}(\xi-t)^{n-m}d\xi \tag{18}$$

The substitution $\zeta=\frac{x-\xi}{x-t}$ in the last integral gives

$$I_x^{\alpha}\left\{\frac{H(x-t)}{\Gamma\left(k-\frac{m-n}{2}+1\right)}(x^2-t^2)^{k+(m-n)/2}(x-t)^{n-m}\right\}=$$

$$=\frac{(x-t)^{\alpha+k-(m-n)/2}}{\Gamma(\alpha)\Gamma\left(k-\frac{m-n}{2}+1\right)}$$

$$\int_0^1\zeta^{\alpha-1}(1-\zeta)^{k-(m-n)/2}\left((x+t)-(x-t)\zeta\right)^{k+(m-n)/2}d\zeta \tag{19}$$

Taking into account the result (16), we rewrite (19) in the form:

$$I_x^{\alpha}\left\{\frac{H(x-t)}{\Gamma\left(k-\frac{m-n}{2}+1\right)}(x^2-t^2)^{k+\frac{m-n}{2}}(x-t)^{n-m}\right\}=$$

$$=2^{k+(m-n)/2}t^k(x-t)^{-m/2}(x+t)^{-n/2}P_k^{m,n}\left(\frac{x}{t}\right)H(x-t) \tag{20}$$

hence we get (13).

Applications

Now we give some examples of applications of the above mentioned results for evaluation of some improper integrals with special functions, and solution of the integral equations.

Following the technique of the proof of Theorem 1 we can easily evaluate the following integrals,

$$\int_0^{\pi/2}(ch(\beta)+sh(\beta)\cos(x))^{k+(m+n)/2}$$

$$\sin^{-2m}(x)\,{}_2F_1(n-m,-k-\frac{m+n}{2};\frac{1}{2}-m;$$

$$\frac{sh^2\left(\frac{\beta}{2}\right)\sin^2(x)}{ch(\beta)+sh(\beta)\cos(x)})\,dx=$$

$$=\sqrt{\pi}\,2^{-n}(ch(\beta)+1)^{\frac{n-m}{2}}sh^m(\beta)$$

$$\Gamma\left(\frac{1}{2}-m\right)P_k^{m,n}(ch(\beta)), \tag{21}$$

$$\left(Re(m)<\frac{1}{2},\ k>\frac{-(m+n)}{2}-1,\ 0<n-m<1\right);$$

$$\int_0^{\infty}x^{-2m}[I_r(bx)\,Y_{-r}(cx)+I_{-r}(bx)\,Y_r(cx)]$$

$${}_2F_3(n-m,-k-\frac{m+n}{2};\frac{1}{2}-m,-n-2k-\frac{1}{2},n-2m+2k+\frac{3}{2};$$

$$\frac{-x^2}{4}(b-c)^2)dx=$$

$$=\frac{-2^{-m/2-3n/2}(|b-c|)^m(b+c)^n\Gamma\left(-n-2k-\frac{1}{2}\right)}{\sqrt{\pi}\,(bc)^{\frac{1+n-m}{2}}\Gamma\left(\frac{1}{2}+m\right)\Gamma^{-1}\left(n-2m+2k+\frac{3}{2}\right)}$$

$$P_k^{m,n}\left(\frac{b^2+c^2}{2bc}\right) \tag{22}$$

$$\left(r=m-n-2k-1,\ |m|<\frac{1}{2},\ n>m,\ k<\frac{-(m+n)}{2},\right.$$

$$\left.n-2m+2k+\frac{3}{2}>0;\ b,c>0;\right)$$

$$\int_0^t(t-x)^{-m-1/2}\left[\frac{[\sqrt{x+2}+\sqrt{x}\,]^s+[\sqrt{x+2}-\sqrt{x}\,]^s}{\sqrt{x(x+2)}}\right]*$$

$$\int_0^t{}_2F_0\left(n-m,-k-\frac{m+n}{2};\frac{t-x}{2(t+2)}\right)dx= \tag{23}$$

$$=\sqrt{\pi}\,2^{1-n}t^{-m/2}(t+2)^{n/2-m}P_k^{m,n}(t+1),$$

$$\left[s=k-m+n+1,\ -k-\frac{m+n}{2}>0,\ n>m\right].$$

Let us consider the following integral equation:

$$\int_a^x(x-t)^{-m/2}(x+t)^{-n/2}P_k^{m,n}\left(\frac{x}{t}\right)g(t)dt=f(x),(a<x\le\beta) \tag{24}$$

Here $g(x)$ is unknown function, absolutely, continuous on $[a;b]$, $g(a)=0$, $f(x)$ is the given function, absolutely continuous on $[a;b]$.

Taking in account (20), we reduce (24) to an equivalent integral equation:

$$\int_a^b I_x^{\alpha} \left\{ \frac{H(x-t)}{\Gamma\left(k-\frac{m-n}{2}+1\right)} (x^2-t^2)^{k+(m-n)/2} (x-t)^{n-m} 2^{-k-(m-n)/2} t^{-k} \right\} g(t)dt = f(x) \tag{25}$$

Interchanging the orders of integration which is permisible by Fubini's Theorem), using the inverse operator $I_x^{-\alpha}$ we rewrite (25) in the following form:

$$\int_a^x (x+t)^l (x-t)^q \, \tilde{g}(t)dt = F(x) \tag{26}$$

where

$$l = k+\frac{m-n}{2},\; q = k-\frac{m-n}{2};\; \tilde{g}(t) = 2^{-k-(m-n)/2} t^{-k} g(t);$$

$$F(x) = I_x^{-\alpha} f(x)$$

For special values of l and q, we can apply the Laplace transform to (26).

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