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Triple Dirichlet Average and Fractional Derivative

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Abstract

The object of the present paper is to establish some results of triple Dirichlet average, using fractional calculus.

Key Words: Dirichlet average, Fractional Derivatives.

El Promedio Triple Dirichlet y la Derivada Fraccional.

Resumen

El objeto del presente trabajo es establecer algunos resultados del Promedio Triple de Dirichlet, usando cálculo fraccional.

Palabras Claves: Promedio Dirichlet, derivada fraccional.

Introduction

Carlson [1] has defined Dirichlet averages of functions, which denote a certain kind of integral averages with respect to a Dirichlet measure. Recently Gupta and Agrawal [2] have shown that the double Dirichlet average is equivalent to fractional derivative of two variables and it can be transformed to Appell functions of two variables.

We mention some relevant definitions.

Dirichlet measure:

Let $b \in C^k_>$: $k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b defined on E by

$$d\mu_b = \frac{1}{B(b)} u_1^{b_1-1} \ldots u_{k-1}^{b_{k-1}^{-1}}$$

$$(1-u_1 \ldots u_{k-1})^{b_k-1} du_1 \ldots du_{k-1} \qquad (1.1)$$

where

$$B(b) = \frac{\Gamma(b_1)\ldots\Gamma(b_k)}{\Gamma(b_1+\ldots+b_k)} \qquad (1.2)$$

Dirichlet average:

Let Ω be a convex set in C, let $z = (z_1,\ldots,z_k) \in \Omega^k$, $k \geq 2$ and let u.z be the convex combination of $z_1,\ldots,z_k$. Let f be a measurable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}. Then the definition is,

$$F(b,z) = \int_E f(u.z) d\mu_b(u) \qquad (1.3)$$

where F is the Dirichlet average of f with variables $z = (z_1,\ldots z_k)$ and parameters $b = (b_1,\ldots\ldots,b_k)$.

If $k = 1$; we have $F(b,z) = f(z)$.

Dirichlet average of x^n:

Let $n \in N$ and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}, $k \geq 2$. For every $z \in C^k$ define

$$R_n(b,z) = \int_E (u.z)^n d\mu_b(u); \qquad (1.4)$$

particularly when k= 2, we obtain,

$$R_n(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\times$$

$$\int_0^1[ux+(1-u)y]^n u^{\beta-1}(1-u)^{\beta'-1}du \quad (1.5)$$

where β, β' have positive real parts and x, y are unrestricted.

Dirichlet average of x^t:

Let μ_b be a Dirichlet measure on the standard simplex $E\subset R^{k-1}$, $k\geq 2$. Let H be half plane in C - { 0 }.

Let Ω = H if t ∈ C - N, but if t ∈ N, let Ω = C. For every $z\in\Omega^k$ define

$$R_t(b,z)=\int_E (u.z)^t\, d\mu_b(u) \quad (1.6)$$

If k= 1, define $R_t(b,z)=z^t$

where alternative notation is $R(-t, b, z) = R_t(b,z)$.

Triple average of function of one variable:

Let z be species with complex elements z_{ijk}.

Let $u = (u_1,...,u_l)$ be an ordered l-tuple of real non-negative weights with $\sum u_i = 1$ and $v = (v_1,...,v_m)$ be an ordered m-tuple of real nonnegative weights $\sum v_j = 1$ and similarly $w = (w_1,...,w_n)$ be an ordered n-tuple of real nonnegative weights with $\sum w_k = 1$.

We define,

$$u.z.v.w=\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} u_i z_{ijk} v_j w_k \quad (1.7)$$

If z_{ijk} is regarded as a point of the complex plane, all these convex combinations are points, in convex hull, denoted by H(z).

Let $\mu = (\mu_1,...,\mu_l)$ be an ordered l-tuple of complex numbers with positive real parts (Re (μ) > 0) and similarly for $\alpha = (\alpha_1,....,\alpha_m)$ and $\beta = (\beta_1,.....\beta_n)$

Then we define $dm_\mu(u)$, $dm_\alpha(v)$ and $dm_\beta(w)$ as (1.1). Let f be holomorphic on a domain D in the complex plane.

If Re (μ) > 0, Re (α) > 0, Re (β) > 0 and H (z) ⊂ D, we define,

$$F(\mu, z, \alpha, \beta) =$$
$$\iiint f(u.z.v.w)^t\, dm_\mu(u)\, dm_\alpha(v)\, dm_\beta(w) \quad (1.8)$$

Corresponding to the particular function z^t we define,

$$R_t(\mu, z, \alpha, \beta) =$$
$$\iiint (u.z.v.w)^t\, dm_\mu(u)\, dm_\alpha(v)\, dm_\beta(w) \quad (1.9)$$

Fractional derivative:

One of the simplest definitions of an integral of fractional order is based on an integral transform, called Riemann-Liouville operator of fractional integration, explained in Oldham-Spanier [4] is

$$D_{0,z}^{-\nu}[f(z)]=\frac{1}{\Gamma(\nu)}\int_0^z f(t)(z-t)^{\nu-1}dt \quad (1.10)$$

$$\mathrm{Re}(\nu)>0$$

For further account of the subject, one may refer Nishimoto [3].

Main Results

Following results, expressing the equivalence of triple Dirichlet average and fractional derivatives, have been established.

$$R_n(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=$$
$$\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}R_n(\mu,\mu';x,y)=$$
$$\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}(y-x)^{1-\mu-\mu'}\frac{\Gamma(\mu+\mu')}{\Gamma(\mu)}D_{y-x}^{-\mu'}x^n(y-x)^{\mu-1}$$

(2.1)

$$R_{-\nu}(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')\Gamma(\alpha+\alpha')\Gamma(\beta+\beta')}{\Gamma(\mu)\Gamma(\alpha)\Gamma(\beta)}$$

$$\times(1-d)^{1-\mu-\mu'}(1-f)^{1-\alpha-\alpha'}(1-g)^{1-\beta-\beta'}$$

$$\times D_{1-g}^{-\beta'}D_{1-f}^{-\alpha'}D_{1-d}^{-\mu'}[gfd]^{-\nu}(1-g)^{\beta-1}(1-f)^{\alpha-1}(1-d)^{\mu-1}$$

(2.2)

$$R_{-\nu}(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')\Gamma(\alpha+\alpha')\Gamma(\beta+\beta')}{\Gamma(\mu)\Gamma(\alpha)\Gamma(\beta)}$$

$$\times (1-d)^{1-\mu-\mu'} (1-f)^{1-\alpha-\alpha'} (1-g)^{1-\beta-\beta'}$$

$$\times D_{1-g}^{-\beta'} D_{1-f}^{-\alpha'} D_{1-d}^{-\mu'} [d+f+g-2]^{-\nu}(1-d)^{\mu-1}(1-f)^{\alpha-1}(1-g)^{\beta-1} \quad (2.3)$$

Proofs

Let us consider the triple average for (= m = n = 2) of x^{-t} from (1.9):

$$R_t(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \int_0^1\int_0^1\int_0^1 [u.z.v.w]^t \, dm_{(\mu,\mu')}(u) \times dm_{(\alpha,\alpha')}(v)\, dm_{(\beta,\beta')}(w), \quad (2.4)$$

where $Re(\mu) > 0$, $Re(\mu') > 0$, $Re(\alpha) > 0$,

$Re(\alpha') > 0$, $Re(\beta) > 0$, $Re(\beta') > 0$,

and

$$u.z.v.w. = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} u_i\, z_{ijk}\, v_j\, w_k$$

$$= [u_1z_{111}v_1w_1 + u_1z_{112}v_1w_2 + u_1z_{121}v_2w_1 + u_1z_{122}v_2w_2 + u_2z_{211}v_1w_1 + u_2z_{212}v_1w_2 + u_2z_{221}v_2w_1 + u_2z_{222}v_2w_2]$$

Assume in first species

$z_{111} = a$, $z_{112} = b$, $z_{121} = c$, $z_{122} = d$

and in second species

$z_{211} = e$, $z_{212} = f$, $z_{221} = g$, $z_{222} = h$

and let also,

$u_1 = u$; $u_2 = 1 - u$

$v_1 = v$; $v_2 = 1 - v$

$w_1 = w$; $w_2 = 1 - w$,

such that

$$[u.z.v.w] = [uvw(a - b - c + d - e + f + g - h) + uv(b - d - f + h) + vw(e - f - g + h) + wu(c - d - g + h) + u(d - h) + v(f - h) + w(g - h) + h]$$

and

$$dm_{(\mu,\mu')}(u) = \frac{\Gamma(\mu+\mu')}{\Gamma(\mu)\,\Gamma(\mu')} u^{\mu-1}(1-u)^{\mu'-1}du,$$

$$dm_{(\alpha,\alpha')}(v) = \frac{\Gamma(\alpha+\alpha')}{\Gamma(\alpha)\,\Gamma(\alpha')} v^{\alpha-1}(1-v)^{\alpha'-1}dv,$$

$$dm_{(\beta,\beta')}(w) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\,\Gamma(\beta')} w^{\beta-1}(1-w)^{\beta'-1}dw.$$

Putting these values in (2.4), we get,

$$R_t(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu+\mu')\Gamma(\alpha+\alpha')\Gamma(\beta+\beta')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\alpha)\Gamma(\alpha')\Gamma(\beta)\Gamma(\beta')}$$

$$\times\int_0^1\int_0^1\int_0^1 [uvw(a - b - c + d - e + f + g - h) + uv(b-d-f+h) + vw(e-g-f+h) + wu(c-d-g+h) + u(d - h) + v(f - h) + w(g - h) + h]^t$$

$$\times u^{\mu-1}(1-u)^{\mu'-1} v^{\alpha-1}(1-v)^{\alpha'-1} w^{\beta-1}(1-w)^{\beta'-1} du\, dv\, dw \quad (2.5)$$

In order to obtain the fractional derivative equivalent to the above integral, consider the following cases:

Case I: If $a = x$; $e = y$; $b = c = d = f = g = h = 0$ and $t = n$ in (2.5), then we have

$$R_n(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu+\mu')\Gamma(\alpha+\alpha')\Gamma(\beta+\beta')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\alpha)\Gamma(\alpha')\Gamma(\beta)\Gamma(\beta')}$$

$$\times \int_0^1\int_0^1\int_0^1 [uvw(x-y) + vwy]^n u^{\mu-1}(1-u)^{\mu'-1}$$

$$\times v^{\alpha-1}(1-v)^{\alpha'-1} w^{\beta-1}(1-w)^{\beta'-1} du\, dv\, dw.$$

Using the definition of Beta function and due to suitable adjustments, we arrive at

$$R_n(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n} \cdot \frac{\Gamma(\mu+\mu')}{\Gamma(\mu)\Gamma(\mu')}$$

$$\times \int_0^1 [ux+(1-u)y]^n u^{\mu-1} (1-u)^{\mu'-1} du$$

$$= \frac{(\alpha)_n (\beta)_n}{(\alpha+\alpha')_n (\beta+\beta')_n} R_n (\mu, \mu' ; x, y)$$

using [4] we arrive at the result

$$R_n (\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{(\alpha)_n (\beta)_n}{(\alpha+\alpha')_n (\beta+\beta')_n} \cdot \frac{\Gamma(\mu+\mu')}{\Gamma(\mu)}$$

$$\times \ (y-x)^{1-\mu-\mu'} D^{-\mu'} x^n (y-x)^{\mu-1}$$

Case II: If a = dfg; b = df; c= dg; e= fg; h = 1 and t = - v in (2.5) we get

$$R_{-v} (\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu+\mu')\Gamma(\alpha+\alpha')\Gamma(\beta+\beta')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\alpha)\Gamma(\alpha')\Gamma(\beta)\Gamma(\beta')}$$

$$\times \int_0^1\int_0^1\int_0^1 [uvw(dfg - df - fg - dg + d + f + g - 1)$$

$$+ uv (df - f - d + 1) + vw(fg - f - g + 1)$$

$$+ wu(dg-d-g+1) + u(d-1) + v(f-1) + w(g-1) + 1]^{-v}$$

$$\times u^{\mu-1}(1-u)^{\mu'-1} v^{\alpha-1}(1-v)^{\alpha'-1} w^{\beta-1} (1-w)^{\beta'-1} du\, dv\, dw.$$

Further, on suitable adjustments of terms, we have

$$R_{v} (\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu+\mu')\Gamma(\alpha+\alpha')\Gamma(\beta+\beta')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\alpha)\Gamma(\alpha')\Gamma(\beta)\Gamma(\beta')}$$

$$\times \int_0^1\int_0^1\int_0^1 [(1-u(1-d)) (1-v(1-f)) (1-w(1-g))]^{-v}$$

$$\times u^{\mu-1}(1-u)^{\mu'-1} v^{\alpha-1}(1-v)^{\alpha'-1} w^{\beta-1} (1-w)^{\beta'-1} du\, dv\, dw.$$

Putting $u = \frac{p}{1-d}$, $v = \frac{q}{1-f}$, $w = \frac{r}{1-g}$,

we obtain,

$$R_{v} (\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu+\mu')\Gamma(\alpha+\alpha')\Gamma(\beta+\beta')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\alpha)\Gamma(\alpha')\Gamma(\beta)\Gamma(\beta')}$$

$$\times (1-d)^{1-\mu-\mu'} (1-f)^{1-\alpha-\alpha'} (1-g)^{1-\beta-\beta'}$$

$$\times \int_0^{1-g}\int_0^{1-f}\int_0^{1-d} [(1-p)(1-q)(1-r)]^{-v} r^{\beta-1}(1-g-r)^{\beta'-1}$$

$$\times q^{\alpha-1} (1-f-q)^{\alpha'-1} p^{\mu-1} (1-d-p)^{\mu'-1} dp\, dq\, dr.$$

Now using definition of fractional derivative (1.10), and on obvious adjustments of terms, we get

$$R_{-v} (\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu+\mu')\Gamma(\alpha+\alpha')\Gamma(\beta+\beta')}{\Gamma(\mu)\Gamma(\alpha)\Gamma(\beta)}$$

$$\times (1-d)^{1-\mu-\mu'} (1-f)^{1-\alpha-\alpha'} (1-g)^{1-\beta-\beta'}$$

$$\times D_{1-d}^{-\mu'} D_{1-f}^{-\alpha'} D_{1-g}^{-\beta'} [dfg]^{-v}(1-g)^{\beta-1}(1-f)^{\alpha-1}(1-d)^{\mu-1}$$

which completes the analysis for (2.2).

Case III: If a = d + f + g - 2, b = d + f - 1, c = d + g - 1, e = f + g - 1, t = -v and h = 1 in (2.5) then we get,

$$R_{v} (\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu+\mu')\Gamma(\alpha+\alpha')\Gamma(\beta+\beta')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\alpha)\Gamma(\alpha')\Gamma(\beta)\Gamma(\beta')}$$

$$\times \int_0^1\int_0^1\int_0^1 [1-u(1-d) - v(1-f) - w(1-g)]^{-v}$$

$$\times u^{\mu-1}(1-u)^{\mu'-1} v^{\alpha-1}(1-v)^{\alpha'-1} w^{\beta-1} (1-w)^{\beta'-1} du\, dv\, dw.$$

On putting $u = \frac{p}{1-d}$, $v = \frac{q}{1-f}$, and $w = \frac{r}{1-g}$, we get

$$R_{v} (\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu+\mu')\Gamma(\alpha+\alpha')\Gamma(\beta+\beta')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\alpha)\Gamma(\alpha')\Gamma(\beta)\Gamma(\beta')}$$

$$\times (1-d)^{1-\mu-\mu'} (1-f)^{1-\alpha-\alpha'} (1-g)^{1-\beta-\beta'}$$

$$\times \int_0^{1-g}\int_0^{1-f}\int_0^{1-d} [1-p-q-r]^{-v} r^{\beta-1} (1-g-r)^{\beta'-1}$$

$$\times q^{\alpha-1} (1-f-q)^{\alpha'-1} p^{\mu-1} (1-d-p)^{\mu'-1} dp\, dq\, dr.$$

Now using the definition of fractional derivative (1.10), we have,

$$R_{-v} (\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu+\mu')\Gamma(\alpha+\alpha')\Gamma(\beta+\beta')}{\Gamma(\mu)\Gamma(\alpha)\Gamma(\beta)}$$

$$\times (1-d)^{1-\mu-\mu'} (1-f)^{1-\alpha-\alpha'} (1-g)^{1-\beta-\beta'}$$

$$\times D_{1-d}^{-\mu'} D_{1-f}^{-\alpha'} D_{1-g}^{-\beta'} [d+f+g-2]^{-v}$$

which completes the analysis of (2.3).

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