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ON THE FRACTIONAL CALCULUS OPERATOR INVOLVING GAUSS'S SERIES AND ITS APPLICATION TO CERTAIN STATISTICAL DISTRIBUTIONS

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ABSTRACT

This paper is dealing with a generalized fractional calculus introduced by the first author. The images of some elementary functions under the operators of this calculus are presented in terms of multiple hypergeometric functions. These results are applied to a certain statistical distribution and a generalized Dirichlet distributions. Further results concerning these distributions and known particular cases are also mentioned.

RESUMEN

Este trabajo trata de un cálculo fraccional generalizado introducido por el primer autor . Las imágenes de algunas funciones elementales bajo los operadores de este cálculo son presentadas en términos de funciones hipergeométricas múltiples. Estos resultados son aplicados a ciertas distribuciones estadísticas y a distribuciones generalizadas de Dirichlet. Además se mencionan también resultados relativos a estas distribuciones y casos particulares conocidos.

1. INTRODUCTION

For α, β and η any complex numbers with $\mathrm{Re}(\alpha) > 0$, we define the fractional integral of a function $f(x)$ by (see [11])

$$I_{0,x}^{\alpha,\beta,\eta}f(x) =$$

$$\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)f(t)dt. \qquad (1.1)$$

The fractional derivative of $f(x)$ for $\mathrm{Re}(\alpha) \le 0$ is given bv

$$I_{0,x}^{\alpha,\beta,\eta}f(x) = \frac{d^n}{dx^n}I_{0,x}^{\alpha+n,\beta-n,\eta-n}f(x), \qquad (1.2)$$

provided that n is a positive integer such that $-\mathrm{Re}(\alpha) < n \le -\mathrm{Re}(\alpha) + 1$.

The Gauss hypergeometric function $F(a,b;c;z)$ appearing in (1.1) is a special case of the generalized hypergeometric function

$${}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;z) = \sum_{m=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_m}{\prod_{j=1}^{q}(b_j)_m}\frac{z^m}{m!} \qquad (1.3)$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad \text{for } n = \text{nonnegative integer},$$

is the Pochhammer symbol and $F(a,b;c;z) = {}_2F_1(a,b;c;z)$, i.e. $p=2$, $q=1$.

Special cases of (1.1) yield the Riemann-Liouville and Erdélyi-Kober fractional calculus operators

$$R_{0,x}^{\alpha}f(x) = I_{0,x}^{\alpha,-\alpha,\eta}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \text{ and}$$

$$E_{0,x}^{\alpha,\eta}f(x) = I_{0,x}^{\alpha,0,\eta}f(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\eta}f(t)dt,$$

respectively, (cf. [11])

The fractional calculus operator $I_{0,x}^{\alpha,\beta,\eta}$ of integral (1.1) and of differential (1.2) was introduced by the first author [11], and later on applied to various problems in analysis, including the Euler-Darboux equation [12], [19], the theory of univalent functions [20], [8], [16], the potential theory [5], the theory of integral transforms [21], [4], [15] and so on. A calculus for theoperators $I_{0,x}^{\alpha,\beta,\eta}$ and its adjoint form $J_{x,\infty}^{\alpha,\beta,\eta}$ was also discussed in [13] on the space $F_{p,\mu}$ defined by A.C. McBride and on its dual space of generalized functions $F'_{p,\mu}$.

The present paper is intended to apply our fractional calculus operator to certain statistical distributions. We first mention several formulas giving the fractional differintegrals calculus of some functions in terms of multiple hypergeometric functions (1.5) and (1.9), below. These results are then applied to our problem of statistical distribution. Further results concerning these distributions are obtained, and known particular cases are also mentioned. Incidentally, the fractional calculus operators have been used in statistical problems, cf. e.g.[10] and [22].

In this work we shall use the generalized Kampé de Fériet function (see[18, p.38]) of the form

$$F_{l:m_1;\cdots;m_n}^{p:q_1;\cdots;q_n}\left[\begin{array}{l}(a_p):(b'_{q_1});\ \cdots;(b^{(n)}_{q_n});\\ \\ (\alpha_l):(\beta'_{m_1});\ \cdots;(\beta^{(n)}_{m_n});\end{array} x_1,\cdots,x_n\right] = \sum_{s_1,\cdots,s_n=0}^{\infty}\Lambda(s_1,\cdots,s_n)\frac{x_1^{s_1}}{s_1!}\cdots\frac{x_n^{s_n}}{s_n!},$$

where

$$\Lambda(s_1,\cdots,s_n) = \frac{\prod_{j=1}^{p}(a_j)_{s_1+\cdots+s_n}\prod_{j=1}^{q_1}(b'_j)_{s_1}\cdots\prod_{j=1}^{q_n}(b_j^{(n)})_{s_n}}{\prod_{j=1}^{l}(\alpha_j)_{s_1+\cdots+s_n}\prod_{j=1}^{m_1}(\beta'_j)_{s_1}\cdots\prod_{j=1}^{m_n}(\beta_j^{(n)})_{s_n}}, \qquad (1.4)$$

(a_p) abbreviates the array of p parameters $a_1,\ldots,a_p$, etc. Especially, we deal with two such series:

$$_{(n)}F_{2:0}^{2:1} = F_{2:0;\cdots;0}^{2:1;\cdots;1}\left[\begin{array}{l}a_1,a_2:\ b_1;\ \cdots;\ b_n;\\ \\ c_1,c_2:\ -;\ \cdots;\ -;\end{array} x_1,\cdots,x_n\right] \qquad (1.5)$$

for max $\{|x_1|,\ldots,|x_n|\} < 1$, and

$$_{(n)}F_{1:0;1}^{1:1;2} = F_{1:0;\cdots;0;1}^{1:1;\cdots;1;2}\left[\begin{array}{l}a:\ b_1;\ \cdots;\ b_n;\ c_1,c_2;\\ \\ d:\ -;\ \cdots;\ -;\quad e;\end{array} x_1,\cdots,x_n\right] \qquad (1.6)$$

for max $\{|x_1|,\ldots,|x_n|\} < 1$.

The series (1.5) is a generalization of the Lauricella fourth seires [6]

$$F_D^{(n)}[a,b_1,\cdots,b_n;c;x_1,\cdots,x_n] = \sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \qquad (1.7)$$

for max $\{|x_1|,\ldots,|x_n|\} < 1$.

It is known (cf.[18, p.35]) that the confluent series of $F_D^{(n)}$ has the form

$$\Phi_D^{(n)}[a,b_1,\cdots,b_{n-1};c;x_1,\cdots,x_n] = \sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}}{(c)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \qquad (1.8)$$

and its generalization relating to the series (1.5) is recognized to be

$$
{}_{(n)}\Phi^{2:1}_{2:0}\left[\begin{matrix} a_1, a_2: & b_1; & \cdots; & b_{n-1}; & \\ & & & & x_1, \cdots, x_n \\ c_1, c_2: & -; & \cdots; & -; & \end{matrix}\right]
$$

$$
= \sum_{m_1,\cdots,m_n=0}^{\infty} \frac{(a_1)_{m_1+\cdots+m_n}(a_2)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_{n-1})_{m_{n-1}}}{(c_1)_{m_1+\cdots+m_n}(c_2)_{m_1+\cdots+m_n}} \frac{x_1{}^{m_1}}{m_1!}\cdots\frac{x_n{}^{m_n}}{m_n!}. \tag{1.9}
$$

2. FRACTIONAL CALCULUS OF ELEMENTARY FUNCTIONS

Throughout this paper we will denote $\kappa \in A_{\beta,\eta}$ if complex numbers κ, β,η satisfy the inequality

$$
\mathrm{Re}(\kappa) > \max[0, \mathrm{Re}(\beta - \eta)] - 1. \tag{2.1}
$$

LEMMA 1. If $\mathrm{Re}(\alpha) > 0$, $\kappa \in A_{\beta,\eta}$, *a is a positive number,* c *is a complex number and*

$$
\left|\frac{x}{x+a}\right| < 1, \text{ then}
$$

$$
I^{\alpha,\beta,\eta}_{0,x}\left\{x^{\kappa}(x+a)^{-c}\right\} = \frac{\Gamma(1+\kappa)\Gamma(1+\kappa-\beta+\eta)}{\Gamma(1+\kappa-\beta)\Gamma(1+\kappa+\alpha+\eta)} x^{\kappa-\beta}(x+a)^{-c}
$$

$$
F^{2:2;1}_{2:1;0}\left[\begin{matrix} \alpha, c: & \alpha+\beta, -\eta; & 1+\kappa-\beta+\eta; & \frac{x}{x+a}, \frac{x}{x+a} \\ 1+\kappa-\beta, 1+\kappa+\alpha+\eta: & \alpha; & -; & \end{matrix}\right] \tag{2.2}
$$

PROOF. Let us substitute t by $x(1-v)$ in the integral

$$
I^{\alpha,\beta,\eta}_{0,x}\left\{x^{\kappa}(x+a)^{-c}\right\}
$$

$$
= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\kappa}(t+a)^{-c} F\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) dt, \tag{2.3}
$$

express the Gauss series by a series expansion, and integrate term by term. Thus, (2.3) can be written in the form

$$
I^{\alpha,\beta,\eta}_{0,x}\left\{x^{\kappa}(x+a)^{-c}\right\} \frac{\Gamma(\kappa+1)}{\Gamma(\alpha+\kappa+1)} x^{\kappa-\beta}(x+a)^{-c}
$$

$$
\cdot \sum_{m=0}^{\infty} \frac{(\alpha)_m(c)_m}{(\alpha+\kappa+1)_m m!}
$$

$$
{}_3F_2\left(\alpha+m, \alpha+\beta, -\eta; \alpha, \alpha+\kappa+1+m; 1\right)\left(\frac{x}{x+a}\right)^m
$$

Then an application of the formula [9,(7.4.4.9)]:

$$
{}_3F_2\left(a, b, c; d, a-n; 1\right) =
$$

$$
\frac{\Gamma(d)\Gamma(d-b-c)}{\Gamma(d-b)\Gamma(d-c)} {}_3F_2\left(-n, b, c; b+c-d+1, a-n; 1\right)
$$

for $a \neq 0, -1, -2, \ldots$ and $\mathrm{Re}(d-b-c) > n$ with nonnegative integer n, implies the formula (2.2), where the relation

$$
(a-n-k)_n = (-1)^n \frac{(1-a)_{n+k}}{(1-a)_k}
$$

is invoked.

REMARK 1. When c=0 in Lemma 1, we receive the following known formula due to Saigo and Raina [14, p.16, Lemma 1] without any restriction on the order α of the fractional integration or differentiation:

$$
I^{\alpha,\beta,\eta}_{0,x} x^{\kappa} = \frac{\Gamma(1+\kappa)\Gamma(1+\kappa-\beta+\eta)}{\Gamma(1+\kappa-\beta)\Gamma(1+\kappa+\alpha+\eta)} x^{\kappa-\beta}. \tag{2.4}
$$

LEMMA 2. If $\kappa \in A_{\beta,\eta}$, a_j *and* $c_j (j = 1,\ldots,r)$ *are complex numbers, and* max $[|a_1 x|,\ldots,|a_r x|] < 1$, *then*

$$I_{0,x}^{\alpha,\beta,\eta}\left\{x^{\kappa}\prod_{j=1}^{r}(1-a_jx)^{-c_j}\right\}$$

$$=\frac{\Gamma(1+\kappa)\Gamma(1+\kappa-\beta+\eta)}{\Gamma(1+\kappa-\beta)\Gamma(1+\kappa+\alpha+\eta)}x^{\kappa-\beta} \tag{2.5}$$

$$\cdot\, {}_{(r)}F_{2:0}^{2:1}\left[\begin{matrix}1+\kappa,1+\kappa-\beta+\eta: & c_1; & \cdots; & c_r; \\ 1+\kappa-\beta,1+\kappa+\alpha+\eta: & -; & \cdots; & -;\end{matrix}\; a_1x,\cdots,a_rx\right].$$

PROOF. If we use the expansion for $\prod_{j=1}^{r}(1-a_jx)^{-c_j}$, and change the order of summation and integration in L.H.S. of (2.5), we obtain:

L.H.S. =

$$\sum_{m_1,\cdots,m_r=0}^{\infty}\frac{(c_1)_{m_1}\cdots(c_r)_{m_r}}{m_1!\cdots m_r!}a_1^{m_1}\cdots a_r^{m_r}I_{0,x}^{\alpha,\beta,\eta}x^{\kappa+m_1+\cdots+m_r}.$$

Invoking the known result (2.4) and then interpreting with the aid of definition (1.5), we arrive at (2.5).

REMARK 2. A known special case ($\beta=-\alpha$) of (2.5) is the following formula

$$R_{0,x}^{\alpha}\left\{x^{\kappa}\prod_{j=1}^{r}(1-a_jx)^{-c_j}\right\}=\frac{\Gamma(1+\kappa)}{\Gamma(1+\kappa+\alpha)}x^{\kappa+\alpha}$$

$$F_D^{(r)}[1+\kappa,c_1,\cdots,c_r;1+\kappa+\alpha;a_1x,\cdots,a_rx] \tag{2.6}$$

valid for all values of α provided that $\max[|a_1x|,\ldots,|a_rx|]<1$, which is recorded in Srivastava and Goyal [17, p.649, Eqn.(3.6)].

By using Lemma 4, [14] we obtain:

LEMMA 3. *If* $k\in A_{\beta,\eta}$, $a_j(j=1,\ldots,r)$ *and* $c_j(j=1,\ldots,r-1)$ *are complex numbers, and* $\max[|a_1x|,\ldots,|a_{r-1}x|]<1$ *then*

$$I_{0,x}^{\alpha,\beta,\eta}\left\{x^{\kappa}e^{a_rx}\prod_{j=1}^{r-1}(1-a_jx)^{-c_j}\right\}$$

$$=\frac{\Gamma(1+\kappa)\Gamma(1+\kappa-\beta+\eta)}{\Gamma(1+\kappa-\beta)\Gamma(1+\kappa+\alpha+\eta)}x^{\kappa-\beta} \tag{2.7}$$

$$\cdot\, {}_{(r)}\Phi_{2:0}^{2:1}\left[\begin{matrix}1+\kappa,1+\kappa-\beta+\eta: & c_1; & \cdots; & c_{r-1}; \\ 1+\kappa-\beta,1+\kappa+\alpha+\eta: & -; & \cdots; & -;\end{matrix}\; a_1x,\cdots,a_{r-1}x,a_rx\right].$$

REMARK 3. Another known result of Srivastava and Goyal [17, p.649, Eqn.(3.7)]

$$R_{0,x}^{\alpha}\left\{x^{\kappa}e^{a_rx}\prod_{j=1}^{r-1}(1-a_jx)^{-c_j}\right\} \tag{2.8}$$

$$=\frac{\Gamma(1+\kappa)}{\Gamma(1+\kappa+\alpha)}x^{\kappa+\alpha}\Phi_D^{(r)}[1+\kappa,c_1,\cdots,c_{r-1};1+\kappa+\alpha;a_1x,\cdots,a_{r-1}x,a_rx],$$

provided that $\max[|a_1x|,\ldots,|a_rx|]<1$, and α is arbitrary, follows from (2.7) when $\beta=-\alpha$.

REMARK 4. On replacing a_r by b/λ and c_r by λ and letting $\lambda\to\infty$ in (2.5), and in view of the relations

$$\lim_{|\mu|\to\infty}(\mu)_n\left(\frac{1}{\mu}\right)^n=1,\quad\text{and} \tag{2.9}$$

$$\lim_{\theta\to\infty}\left(1-\frac{x}{\theta}\right)^{-\theta}=e^x, \tag{2.10}$$

we find that (2.7) can be obtained from (2.5) too.

3. STATISTICAL DISTRIBUTION. Let us define a family of distributions having the probability density function (p.d.f.) of the form

$$f(x)=$$

$$\begin{cases}\frac{1}{\Delta}(x-h)^{p-1}(k-x)^{\alpha-1}F\left(\alpha+\beta,-\eta;\alpha;\frac{k-x}{k-h}\right) \\ \quad\cdot\prod_{j=1}^{r}[1-a_j(x-h)]^{-c_j}, & \text{for } h\le x\le k, \\ 0, & \text{elsewhere,}\end{cases} \tag{3.1}$$

where $\alpha > 0, \beta, \eta, a_j, c_j > 0 (j = 1,\dots,r)$ are real numbers with $p-1 \in A_{\beta,\eta}$ and

$$\max_{1 \le j \le r} |a_j(k-h)| < 1,\ h \ne k, \text{ and}$$

$$\Delta = \frac{\Gamma(\alpha)\Gamma(p)\Gamma(p-\beta+\eta)}{\Gamma(p-\beta)\Gamma(p+\alpha+\eta)}(k-h)^{\alpha+p-1} \tag{3.2}$$

$$\cdot {}_{(r)}F^{2:1}_{2:0}\left[\begin{matrix} p, p-\beta+\eta: & c_1; & \cdots; & c_r; \\ p-\beta, p+\alpha+\eta: & -; & \cdots; & -; \end{matrix}\ a_1(k-h), \cdots, a_r(k-h)\right].$$

To verify that (3.1) represents a p.d.f., we notice that

$$\int_{-\infty}^{\infty} f(x)dx = \int_h^k f(x)dx$$

$$= \frac{1}{\Delta}\int_h^k (x-h)^{p-1}(k-x)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; \frac{k-x}{k-h}\right)$$

$$\cdot \prod_{j=1}^{r} [1 - a_j(x-h)]^{-c_j}\, dx. \tag{3.3}$$

Introducing the substitution x for $x - h$, we have

$$\int_h^k f(x)dx = \frac{1}{\Delta}\Gamma(\alpha)u^{\alpha+\beta} I_{0,u}^{\alpha,\beta,\eta}\left\{u^{p-1}\prod_{j=1}^{r}(1-a_j u)^{-c_j}\right\}, \tag{3.4}$$

where $u = k - h$. Making use of Lemma 2, we find that (3.1) is indeed a p.d.f.

4. EXPECTATION OF FUNCTION. For any function g(x), the expectation of g(x) with respect to the p.d.f. f(x) is given by

$$E\{g(x)\} = \int_{-\infty}^{\infty} f(x)g(x)dx. \tag{4.1}$$

Let us consider the function $g(x)$ of the form

$$g(x) = \prod_{j=1}^{s} [1 - b_j(x-h)]^{-d_j} \tag{4.2}$$

with $d_j > 0 (j = 1,\dots,s)$ and $\max_{1 \le j \le s} |b_j(k-h)| < 1$, and let a p.d.f. $f(x)$ be given by (3.1), then we have

$$E\{g(x)\} = \frac{1}{\Delta}\Gamma(\alpha)u^{\alpha+\beta} I_{0,u}^{\alpha,\beta,\eta} \tag{4.3}$$

$$\cdot \left\{u^{p-1}\prod_{j=1}^{r}(1-a_j u)^{-c_j}\prod_{j=1}^{s}(1-b_j u)^{-d_j}\right\},$$

where $u = k-h$. An application of Lemma 2 yields:

$$E\{g(x)\} = \frac{1}{\Delta_1} \tag{4.4}$$

$${}_{(r+s)}F^{2:1}_{2:0}\left[\begin{matrix} p, p-\beta+\eta: & c_1; & \cdots; & c_r; & d_1; & \cdots; & d_s; \\ p-\beta, p+\alpha+\eta: & -; & \cdots; & -; & -; & \cdots; & -; \end{matrix}\right.$$

$$\left. a_1 u, \cdots, a_r u, b_1 u, \cdots, b_s u\right],$$

wnere

$$\Delta_1 = {}_{(r)}F^{2:1}_{2:0}\left[\begin{matrix} p, p-\beta+\eta: & c_1; & \cdots; & c_r; \\ p-\beta, p+\alpha+\eta: & -; & \cdots; & -; \end{matrix}\ a_1 u, \cdots, a_r u\right]$$

and $u = k - h$. (4.5)

5. CUMULATIVE FUNCTION. The cumulative probability function for the distribution function F(t) is given by

$$F(t) = \int_{-\infty}^{t} f(x)dx. \tag{5.1}$$

For the p.d.f. $f(x)$ defined by (3.1) and for $h \le t \le k$, we have

$$F(t) = \int_{-\infty}^{t} f(x)dx$$

$$= \frac{1}{\Delta}\int_{h}^{t} (x-h)^{p-1}(k-x)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; \frac{k-x}{k-h}\right)$$

$$\cdot \prod_{j=1}^{r} [1 - a_j(x-h)]^{-c_j}\, dx$$

$$= \frac{1}{\Delta}\int_{0}^{t-h} z^{p-1}(u-z)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; \frac{u-z}{u}\right) \tag{5.2}$$

$$\cdot \prod_{j=1}^{r} [1 - a_j z]^{-c_j}\, dz,$$

where $u = k - h$. Using the continuation formula of the Gauss function

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-z)$$

$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}$$

$$F(c-a, c-b; c-a-b+1; 1-z),$$

$$|\arg(1-z)| < \pi$$

and the Euler transformation

$$F(a,b;c;z) =$$

$$(1-z)^{c-a-b} F(c-a, c-b; c; z), \quad |\arg(1-z)| < \pi$$

in the resulting two terms on the R.H.S. of (5.2), using the series expansion for the Gauss function and integrating term by term, we are led to the result

$$F(t) = \frac{1}{\Delta_1} \frac{\Gamma(-\beta+\eta)\Gamma(p-\beta)\Gamma(p+\alpha+\eta)}{\Gamma(-\beta)\Gamma(\alpha+\eta)\Gamma(p-\beta+\eta)\Gamma(p+1)} \left(\frac{t-h}{u}\right)^{p}$$

$$\cdot\, {}_{(r+1)}F^{1:1;2}_{1:0;1}\left[\begin{matrix} p: & c_1; & \cdots; & c_r; & -\alpha-\eta+1, \beta+1; \\ p+1: & -; & \cdots; & -; & \beta-\eta+1; \end{matrix}\right.$$

$$\left. a_1(t-h), \cdots, a_r(t-h), \frac{t-h}{u} \right]$$

$$+ \frac{1}{\Delta_1} \frac{\Gamma(\beta-\eta)\Gamma(p-\beta)\Gamma(p+\alpha+\eta)}{\Gamma(-\eta)\Gamma(\alpha+\beta)\Gamma(p-\beta+\eta+1)\Gamma(p)} \left(\frac{t-h}{u}\right)^{p-\beta+\eta}$$

$$\cdot\, {}_{(r+1)}F^{1:1;2}_{1:0;1}\left[\begin{matrix} p-\beta+\eta: & c_1; & \cdots; & c_r; -\alpha-\eta+1, \eta+1; \\ p-\beta+\eta+1: & -; & \cdots; & -; \quad -\beta+\eta+1; \end{matrix}\right.$$

$$\left. a_1(t-h), \cdots, a_r(t-h), \frac{t-h}{u} \right] \tag{5.3}$$

in terms of the generalized Kampé de Fériet series (1.6) and by virtue of the integral representation of the Lauricella function (1.7) (see[2]):

$$F_D[a, b_1, \cdots, b_n; c; x_1, \cdots, x_n]$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 v^{a-1}(1-v)^{c-a-1} \prod_{j=1}^{n} (1 - x_j v)^{-b_j} dv,$$

where $u = k - h$ and Δ_1 is defined in (4.5).

For $t < h$, $F(t) = 0$, and for $t > k$, $F(t)$ is recognized to be (5.3) by replacing $(t-h)/u$ with 1.

REMARK 5. The formula (5.3) reduces in the special case to the result given in [2, p.221] in the special case $\beta = -\alpha$, $h = 0, k = 1$.

6. CHARACTERISTIC FUNCTION

The characteristic function $\varphi(t)$ of a random variable x with respect to a p.d.f. $f(x)$ is defined by

$$\varphi(t) = E\{e^{itx}\} = \int_{-\infty}^{\infty} e^{itx} f(x)dx. \tag{6.1}$$

For the p.d.f. $f(x)$ in (3.1), we have

$$\varphi(t) = \frac{1}{\Delta}\int_{h}^{k} e^{itx}(x-h)^{p-1}(k-x)^{\alpha-1} \tag{6.2}$$

$$\cdot\, F\left(\alpha+\beta, -\eta; \alpha; \frac{k-x}{k-h}\right) \prod_{j=1}^{r} [1 - a_j(x-h)]^{-c_j}\, dx.$$

By putting $x - h = z$,(6.2) gives

$$\varphi(t) = \frac{1}{\Delta}\Gamma(\alpha)u^{\alpha+\beta}e^{iht}I_{0,u}^{\alpha,\beta,\eta}\left\{u^{p-1}e^{itu}\prod_{j=1}^{r}(1-a_j u)^{-c_j}\right\}, \tag{6.3}$$

Where $u = k - h$. Now with the help of Lemma 3, we arrive to the following result:

$$\varphi(t) = e^{iht} \cdot \frac{{}_{(r+1)}\Phi_{2:0}^{2:1}\left[\begin{matrix} p, p-\beta+\eta: & c_1; & \cdots; & c_r; \\ p-\beta, p+\alpha+\eta: & -; & \cdots; & -; \end{matrix} a_1u, \cdots, a_ru, itu\right]}{{}_{(r)}F_{2:0}^{2:1}\left[\begin{matrix} p, p-\beta+\eta: & c_1; & \cdots; & c_r; \\ p-\beta, p+\alpha+\eta: & -; & \cdots; & -; \end{matrix} a_1u, \cdots, a_ru\right]}, \tag{6.4}$$

where $u = k - h$.

7. DISTRIBUTION OF RATIO

In this section we find the form of the distribution of the ratio X/Y, where the random variables X and Y have p.d.f. $p(x)$ and $q(x)$, respectively. We set

$$\begin{cases} w = \dfrac{X}{Y}, \quad \text{and} \\ v = \log w, \end{cases} \tag{7.1}$$

so that $v = \log w = \log X - \log Y$. The characteristic function for v is then

$$\varphi_v(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ivt}p(x)q(y)dxdy$$

$$= \int_{-\infty}^{\infty} x^{it}p(x)dx \int_{-\infty}^{\infty} y^{-it}q(y)dy \quad = I_1I_2, \quad \text{say.} \tag{7.2}$$

Let $h \neq k$ and let the p.d.f. $p(x)$ and $q(y)$ be of the type (3.1) defined by

$$p(x) = \begin{cases} \dfrac{1}{\Delta}(x-h)^{p-1}(k-x)^{\alpha-1}F\left(\alpha+\beta, -\eta; \alpha; \dfrac{k-x}{k-h}\right) \\ \qquad \cdot \prod_{j=1}^{r}[1-a_j(x-h)]^{-c_j}, \quad \text{for } h \le x \le k, \\ 0, \quad \text{elsewhere,} \end{cases} \tag{7.3}$$

and

$$q(x) = \begin{cases} \dfrac{1}{\nabla}(y-h)^{q-1}(k-y)^{\lambda-1}F\left(\lambda+\mu, -\rho; \lambda; \dfrac{k-y}{k-h}\right) \\ \qquad \cdot \prod_{j=1}^{s}[1-b_j(y-h)]^{-d_j}, \quad \text{for } h \le y \le k, \\ 0, \quad \text{elsewhere,} \end{cases} \tag{7.4}$$

where $\alpha > 0, \lambda > 0, \beta, \eta, \mu, \rho, a_j, c_j > 0$ $(j = 1,\dots,r)$, $b_i, d_i > 0$ $(i = 1,\dots,s)$ are real numbers with

$$p-1 \in A_{\beta,\eta},\ q-1 \in A_{\mu,\rho},\ \max_{1\le j\le r} |a_j(k-h)| < 1,$$

$\max_{1\le j\le s} |b_j(k-h)| < 1$, Δ is defined by (3.2), and ∇ is given by

$$\nabla = \frac{\Gamma(\lambda)\Gamma(q)\Gamma(q-\mu+\rho)}{\Gamma(q-\mu)\Gamma(q+\lambda+\rho)}(k-h)^{\lambda+q-1} \tag{7.5}$$

$$\cdot {}_{(s)}F_{2:0}^{2:1}\left[\begin{matrix} q, q-\mu+\rho: & d_1; & \cdots; & d_s; \\ q-\mu, q+\lambda+\rho: & -; & \cdots; & -; \end{matrix} b_1(k-h), \cdots, b_s(k-h)\right]$$

We evaluate the integrals I_1 and I_2 occurring in (7.2) when $p(x)$ and $q(y)$ are defined by (7.3) and (7.4). We have

$$I_1 = \frac{1}{\Delta}\int_h^k x^{it}(x-h)^{p-1}(k-x)^{\alpha-1}F\left(\alpha+\beta, -\eta; \alpha; \frac{k-x}{k-h}\right)$$

$$\cdot \prod_{j=1}^{r}[1-a_j(x-h)]^{-c_j}\,dx$$

$$= \frac{1}{\Delta}\int_0^u z^{p-1}(u-z)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; \frac{u-z}{u}\right) \cdot \prod_{j=1}^{r} [1-a_j z]^{-c_j} (h+z)^{it} dz$$

$$= \frac{1}{\Delta}\Gamma(\alpha) u^{\alpha+\beta} I_{0,u}^{\alpha,\beta,\eta}\left\{ u^{p+it-1}\left(1+\frac{h}{u}\right)^{it} \prod_{j=1}^{r} (1-a_j u)^{-c_j}\right\},$$

where, as before, u stands for k-h. Applying suitably Lemma 2 as

$\left|\frac{h}{u}\right| < 1$, $\max\{|a_1 u|, \ldots, |a_r u|\} < 1$, we get

$$I_1 = \frac{\Gamma(p+it)\Gamma(p+it-\beta+\eta)\Gamma(p-\beta)\Gamma(p+\alpha+\eta)}{\Gamma(p+it-\beta)\Gamma(p+it+\alpha+\eta)\Gamma(p)\Gamma(p-\beta+\eta)}(k-h)^{it}$$
$$= \frac{{}_{(r+1)}F^{2:1}_{2:0}\left[\begin{matrix} p+it, p+it-\beta+\eta: & c_1; & \cdots; & c_r; & -it; \\ p+it-\beta, p+it+\alpha+\eta: & -; & \cdots; & -; & -; \end{matrix}\; a_1 u, \cdots, a_r u, -\frac{h}{u}\right]}{{}_{(r)}F^{2:1}_{2:0}\left[\begin{matrix} p, p-\beta+\eta: & c_1; & \cdots; & c_r; \\ p-\beta, p+\alpha+\eta: & -; & \cdots; & -; \end{matrix}\; a_1 u, \cdots, a_r u\right]}. \tag{7.6}$$

Similarly, proceeding as before we evaluate I_2 and it is given by

$$I_2 = \frac{\Gamma(q-it)\Gamma(q-it-\mu+\rho)\Gamma(q-\mu)\Gamma(q+\lambda+\rho)}{\Gamma(q-it-\mu)\Gamma(q-it+\lambda+\rho)\Gamma(q)\Gamma(q-\mu+\rho)}(k-h)^{-it}$$
$$= \frac{{}_{(s+1)}F^{2:1}_{2:0}\left[\begin{matrix} q-it, q-it-\mu+\rho: & d_1; & \cdots; & d_s; & it; \\ q-it-\mu, q-it+\lambda+\rho: & -; & \cdots; & -; & -; \end{matrix}\; b_1 u, \cdots, b_s u, -\frac{h}{u}\right]}{{}_{(s)}F^{2:1}_{2:0}\left[\begin{matrix} q, q-\mu+\rho: & d_1; & \cdots; & d_s; \\ q-\mu, q+\lambda+\rho: & -; & \cdots; & -; \end{matrix}\; b_1 u, \cdots, b_s u\right]} \tag{7.7}$$

In view of (7.2), (7.6) and (7.7), the density function $f_\nu(x)$ for ν follows readily by effecting the inverse Fourier transform, and we get

$$f_v(x) = \frac{L}{2\pi\Delta_1\nabla_1}\int_{-\infty}^{\infty} e^{-itx}\frac{\Gamma(p+it)\Gamma(p+it-\beta+\eta)}{\Gamma(p+it-\beta)\Gamma(p+it+\alpha+\eta)} \cdot \frac{\Gamma(q-it)\Gamma(q-it-\mu+\rho)}{\Gamma(q-it-\mu)\Gamma(q-it+\lambda+\rho)}$$
$$\cdot {}_{(r+1)}F^{2:1}_{2:0}\left[\begin{matrix} p+it, p+it-\beta+\eta: & c_1; & \cdots; & c_r; & -it; \\ p+it-\beta, p+it+\alpha+\eta: & -; & \cdots; & -; & -; \end{matrix}\; a_1 u, \cdots, a_r u, -\frac{h}{u}\right]$$
$$\cdot {}_{(s+1)}F^{2:1}_{2:0}\left[\begin{matrix} q-it, q-it-\mu+\rho: & d_1; & \cdots; & d_s; & it; \\ q-it-\mu, q-it+\lambda+\rho: & -; & \cdots; & -; & -; \end{matrix}\; b_1 u, \cdots, b_s u, -\frac{h}{u}\right] dt, \tag{7.8}$$

where Δ_1 is given by (4.5),

$$\nabla_1 = {}_{(s)}F^{2:1}_{2:0}\left[\begin{matrix} q, q-\mu+\rho: & d_1; & \cdots; & d_s; \\ q-\mu, q+\lambda+\rho: & -; & \cdots; & -; \end{matrix}\; b_1 u, \cdots, b_s u\right] \tag{7.9}$$

and

$$L = \frac{\Gamma(p-\beta)\Gamma(p+\alpha+\eta)\Gamma(q-\mu)\Gamma(q+\lambda+\rho)}{\Gamma(p)\Gamma(p-\beta+\eta)\Gamma(q)\Gamma(q-\mu+\rho)}. \tag{7.10}$$

The equation (7.8) may be rewritten after putting $it = \theta$ as

$$f_v(x) = \frac{L}{\Delta_1\nabla_1}\sum_{m_1,\cdots,m_r,m,n_1,\cdots,n_s,n=0}^{\infty}\frac{(c_1)_{m_1}\cdots(c_r)_{m_r}(d_1)_{n_1}\cdots(d_s)_{n_s}}{m_1!\cdots m_r! m! n_1!\cdots n_s! n!}$$
$$\cdot a_1^{m_1}\cdots a_r^{m_r} b_1^{n_1}\cdots b_s^{n_s}\left(-\frac{h}{k-h}\right)^{m+n}(k-h)^{M+N}$$
$$\cdot \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(m-\theta)\Gamma(q+N+n-\theta)\Gamma(q-\mu+\rho+N+n-\theta)}{\Gamma(\theta)\Gamma(p-\beta+M+m+\theta)\Gamma(p+\alpha+\eta+M+m+\theta)}$$
$$\cdot \frac{\Gamma(n+\theta)\Gamma(p+M+m+\theta)\Gamma(p-\beta+\eta+M+m+\theta)}{\Gamma(-\theta)\Gamma(q-\mu+N+n-\theta)\Gamma(q+\lambda+\rho+N+n-\theta)} e^{-\theta x} d\theta \tag{7.11}$$

involving the multiple series, where, for convenience,

$$M = \sum_{j=1}^{r} m_j, \quad N = \sum_{j=1}^{s} n_j. \tag{7.12}$$

Expressing the contour integral in terms of the Meijer's G-function [7], we get the density function for ratio $w = X/Y$ on replacing e^x by x given by

$$f_w(x) = \frac{L}{\Delta_1\nabla_1}\sum_{m_1,\cdots,m_r,m,n_1,\cdots,n_s,n=0}^{\infty}\frac{(c_1)_{m_1}\cdots(c_r)_{m_r}(d_1)_{n_1}\cdots(d_s)_{n_s}}{m_1!\cdots m_r! m! n_1!\cdots n_s! n!}$$
$$\cdot (a_1(k-h))^{m_1}\cdots(a_r(k-h))^{m_r}(b_1(k-h))^{n_1}\cdots(b_s(k-h))^{n_s}\left(-\frac{h}{k-h}\right)^{m+n} \tag{7.13}$$

$$G_{6,6}^{3,3}\left(\frac{1}{x}\middle| \begin{matrix} 1-n, 1-p-M-m, 1-p+\beta-\eta-M-m, \\ m, q+N+n, q-\mu+\rho+N+n, \end{matrix}\right.$$
$$\left.\begin{matrix} 0, q-\mu+N+n, q+\lambda+\rho+N+n \\ 1, 1-p+\beta-M-m, 1-p-\alpha-\eta-M-m \end{matrix}\right),$$

where L is given by (7.10), and Δ_1 and ∇_1 are given by (4.5) and (7.9), and M,N are the sums of integers defined by (7.12).

REMARK 6. Each of the results stated in Sections 3 ~ 7 above would yield several known or new results. For example, if $\beta = -\alpha$, $k = 1$ and $h = 0$, then in this special case all the results stated in this paper correspond to the results discussed in [2, pp.219-240], after carrying out necessary simplifications.

8. MULTIVARIATE DISTRIBUTION

In this concluding section we present an extended form of the Dirichlet distribution [2, p.222, §7.2.1]. We know that the Liouville's extension of the Dirichlet's result (see[1]) is

$$\int \cdots \int x_1^{p_1-1} \cdots x_r^{p_r-1} \varphi(x_1 + \cdots + x_r) dx_1 \cdots dx_r$$
$$= B(p_1, \cdots, p_r) \int_\alpha^\beta z^{\sum p_i - 1} \varphi(z) dz, \qquad (8.1)$$

for any integrable function $\varphi(z)$, where the integrated region in L.H.S. is $x_i \geq 0$ $(i = 1, \ldots, r)$, $\alpha \leq x_1 + \ldots + x_r \leq \beta$ with $0 < \alpha < \beta$, $\mathrm{Re}(p_i) > 0$, $(i = 1, \ldots, r)$, the sigma denotes a summation with respect to i throught 1 to r, and B means the generalized beta function

$$B(p_1, \cdots, p_r) = \frac{\Gamma(p_1) \cdots \Gamma(p_r)}{\Gamma(p_1 + \cdots + p_r)}.$$

We consider the multivariate density function as

$$f(x_1, \cdots, x_r)$$
$$= \begin{cases} \Omega x_1^{p_1-1} \cdots x_r^{p_r-1} \left(\sum x_i - \alpha\right)^{p-1} \left(\beta - \sum x_i\right)^{\lambda-1} F\left(\lambda+\mu, -\rho; \lambda; \dfrac{\beta - \sum x_i}{\beta - \alpha}\right) \\ \qquad \alpha \leq x_i \leq \beta, x_i \geq 0 \ (i = 1, \cdots, r), \\ 0, \quad \text{elsewhere}, \end{cases} \qquad (8.2)$$

where $p > 0$, $\lambda > 0$, $p_i > 0 (i = 1, \ldots, r)$, μ and ρ are real numbers with $p - 1 \in A_{\mu,\rho}$, and

$$\Omega^{-1} = B(p_1, \cdots, p_r) \frac{\Gamma(\lambda)\Gamma(p)\Gamma(p-\mu+\rho)}{\Gamma(p-\mu)\Gamma(p+\lambda+\rho)} (\beta-\alpha)^{p+\lambda-1} \beta^{\sum p_i - 1}$$
$$\cdot F_{2:1;0}^{2:2;1}\left[\begin{matrix} \lambda, 1-\sum p_i : & \lambda+\mu, -\rho; & p-\mu+\rho; \\ p-\mu, p+\lambda+\rho : & \lambda; & -; \end{matrix} 1 - \frac{\alpha}{\beta}, 1 - \frac{\alpha}{\beta}\right]. \qquad (8.3)$$

To verify that (8.2) represents a p.d.f., we observe that

$$\int \cdots \int f(x_1, \cdots, x_r) dx_1 \cdots dx_r$$
$$= \Omega \int \cdots \int_{x_i \geq 0, \alpha \leq \sum x_i \leq \beta} x_1^{p_1-1} \cdots x_r^{p_r-1} \left(\sum x_i - \alpha\right)^{p-1} \left(\beta - \sum x_i\right)^{\lambda-1}$$
$$\cdot F\left(\lambda+\mu, -\rho; \lambda; \frac{\beta - \sum x_i}{\beta-\alpha}\right) dx_1 \cdots dx_r$$
$$= \Omega B(p_1, \cdots, p_r) \int_\alpha^\beta z^{\sum p_i - 1} (z-\alpha)^{p-1} (\beta - z)^{\lambda-1} F\left(\lambda+\mu, -\rho; \lambda; \frac{\beta - z}{\beta - \alpha}\right) dz$$
$$= \Omega B(p_1, \cdots, p_r) \Gamma(\lambda) u^{\lambda+\mu} I_{0,u}^{\lambda,\mu,\rho} \left\{u^{p-1} (u+\alpha)^{\sum p_i - 1}\right\}, \qquad (8.4)$$

where $u = \beta - \alpha$. Now using Lemma 1 and in view of (8.3), we find that the value of R.H.S. of (8.4) is unity, showing thereby that (8.2) is a p.d.f.

The characteristic function, the distribution function and the expectation of certain functions can be obtained for the multivariate distribution (8.2), we skip the details and end this paper by mentioning that the distribution (8.2) corresponds to the Dirichlet distribution [2, p.222] in the special case when $\alpha = 0$, $\beta = 1$, $p = 1$, and $\mu = -\lambda$.

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REFERENCES

[1] EDWARDS, A.J.: *A treatise on Integral Calculus*, Chelsea Publ., New York, 1954.

[2] EXTON, H.: *Multiple Hypergeometric Functions and Applications*, Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley and Sons Inc.], New York-London-Sydney-Toronto, 1976.

[3] GLAESKE, H.J. and SAIGO, M.: Products of Laplace transforms and fractional integrals on spaces of generalized functions, Math. Japon., (to appear).

[4] KILBAS, A.A.; SAIGO, M. AND JUKH, V.A.: On compositions of generalized fractional calculus operators with the differential operator of axisymmeric potential theory, (submitted for publication).

[5] LAURICELLA, G.: Sulle funzioni ipergeometriche a più variabili, Rend. Circ. Mat. Palermo, 7(1893), 111-158.

[6] MATHAI, A.M. AND SAXENA, R.K.: *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*, Lecture Notes in Math., Vol. 348, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[7] OWA, S.; SAIGO M. AND SRIVASTAVA, H.M.: Some characterization theorems for starlike and and convex functions involving certain fractional integral operator, J. Math. Anal. Appl., 140(1989), 419-426.

[8] PRUDNIKOV, A.P.; BRYCHKOV, YU.A. AND MARICHEV, O.I.: *Integrals and Series, Vol. 3, More Special functions*, Golfdon and Breach Science Publ., New York -Philadelphia- London- Paris- Montreux-Tokyo-Melbourne, 1990.

[9] RAINA, R.K.: The H-function transform and the moments of probability distribution functions of an arbitrary order, *Simon Stevin*, 60(1986), 97-103.

[10] SAIGO, M.: A remark on integral operators involving the Gauss hypergeometric functions, Math. Rep. Kyushu Univ., 11(1978), 135-143.

[11] SAIGO, M.: A certain boundary value problem for the Euler-Darboux equation, Math. Japon, 24(1979), 377-385; II, ibid. 25(1980), 211-220; III, ibid. 26(1981), 103-119.

[12] SAIGO, M: A certain boundary value problem for the Euler-Darboux equation, Math. Japon, 24(1979), 377-385; II, ibid. 25(1980), 211-220; III, ibid. 26(1981), 103-119.

[13] SAIGO, M. and GLAESKE, H.J.: Fractional calculus operators involving the Gauss function in spaces $F_{p,\mu}$, and $F'_{p,\mu}$, Math. Nachr., 147(1990), 285-306.

[14] SAIGO, M. and RAINA, R.K.: Fractional operators associated with a general class of polynomials, Fukuoka Univ. Sci.Rep., 18(1988), 15-22.

[15] SAIGO, M.; SAXENA, R.K. and RAM, J.: Certain properties of the Saigo operator of fractional integration, (in preparation).

[16] SHOHI, N.S.: Distortion theorems involving certain operators of fractional calculus on a class of p-valent functions, Fractional Calculus and Its Appl., 245-252, Proc. Intn. Conf. Nihon Univ. 1989, edited by K. Nishimoto, College of Engng. Nihon Univ., Koriyama, 1990.

[17] SRIVASTAVA, H.M. AND GOYAL, S.P.: Fractional derivatives of the H-function of several variables, J. *Math. Anal. Appl., 112(1985), 641-651.*

[18] SRIVASTAVA, H.M. AND KARLSSON, P.W.: Multiple Gaussian Hypergeometric Series, Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley and Sons Inc.], New York-Chichester-Brisbane-Toronto, 1985.

[19] SRIVASTAVA, H.M. AND SAIGO, M.: Multiplication of fractional calculus operators and boundary value problems involving the Euler-Darboux equation, J.Math. Anal., 121(1987), 325-369.

[20] SRIVASTAVA, H.M.; SAIGO, M. AND OWA, S.: A class of distortion theorems involving certain operators of fractional calculus, J. Math. Anal. Appl., 131(1988), 412-420.

[21] SRIVASTAVA, H.M.; SAIGO, M. AND RAINA, R.K.: Certain results connecting fractional operators with the Laplace transform, Univ. of Victoria Report, No. DM-481-IR, 1-16, 1988.

[22] WOLFE, S.L.: On moments of probability distribution functions, Fractional Calculus and Its Applications, 306-316, Lecture Notes in Math., edited by B. Ross, Vol. 457, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

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NOTE:

It is worth mentioning that generalized fractional calculi whose operators involve the Gauss function have been considered also by: E.R. Love (1967-1974), S.L.Kalla and R.K. Saxena (1969, 1974), S.L. Kalla (1980), A. McBride (1982, 1984), etc. All these calculi are quite special cases of the generalized fractional calculus dealing with G- and H-functions as kernels: Kalla (1969, 1980), Kiryakova (1986-1989) Kalla and Kiryakova (1990). From this point of view, the applications of the results in Section 2 to statistical problems in Sections 3-8 are interesting as showing some of the various applications of this topic, and have practical importance.