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FINITE LAPLACE TRANSFORMATION

ABSTRACT

Over the years several authors have obtained various properties of the Laplace transformation with the integration restricted to a finite interval. We discuss the operational properties and illustrate their applications to a boundary value problem for each an ordinary and a partial differential equation problem. The compatibility conditions which arise from the property that the transform is an entire function play a key rôle in applications to these boundary value problems.

RESUMEN

A través de los años diversos autores han obtenido varias propiedades de la transformada de Laplace con la integración restringída a un intervalo finito. Nosotros discutiremos las propiedades operacionales e ilustraremos sus aplicaciones a un problema de valor de contorno en el caso que se tenga una ecuación diferencial ordinaria ó en drivadas parciales. Las condiciones de compatibilidad las cuales surgen de la propiedad de que la transformada es una función entera juega un papel decisivo en las aplicaciones a estos problemas de valores de contorno.

1. INTRODUCTION

Scattered over the past 50 years there has been occasional work which has appeared in the literature in regard to the *finite Laplace transformation*. For our purposes we take

$$\int_0^L e^{-st} f(t)dt = L_L\{f(t)\} = \hat{f}_L(s)$$

for the defining integral and notation. If we denote the unit step function by u(u(t) = 1 if t>0, u(t) = 0 if t<0),

$$L_L\{f(t)\} = L\{f(t)u(L-t)\}$$

gives us the connection between the finite Laplace transformation and the ordinary Laplace transformation. Many, but not all, of the properties for the finite case are thus inherited from the infinite case.

Necessary and sufficient conditions that a Laplace transform correspond to an original function f(t) which is zero for t>L were given by Doetsch [2]. Further results in regard to the cut-off point were developed by Doetsch [4, pp. 225-232]. Titchmarsh [11] obtained information on the number of zeros of $\hat{f}_L(s)$. The result that $\hat{f}_L(s)$ is an entire function seems to be due to Pincherle [9]; the proof by Landau [8] can also be found in [3, p.145].

Some of the operational properties have been developed by Dunn [5], and similarly by Debnath and Thomas [1]. Applications to problems in differential equations were given by Doetsch [4, pp. 261-2], Dunn [5], and Debnath and Thomas [1]. Ghizetti [7] introduced a more general format for the definition in connection with an application to partial differential equations. He uses

$$\int_{\alpha(x)}^{\beta(x)} e^{qy}u(x,y)dy = \hat{u}(x,q) ,$$

where the variable limits were used in order to study Laplace's equation on the region $0 < x < 1$, $\alpha(x) \le y \le \beta(x)$. Useful tables entries can be found in Roberts and Kaufman [10], especially section 33 which is devoted to "window functions". As will be explained later, inversion tables for the ordinary Laplace transformation are sufficient for boundary value problems on [0,L].

The transform $\hat{f}_L(s)$ is an entire function. This property, the use of which is essential for applications, may be what has stood in the way of wider use of this finite case. Complex analysis is usually avoided in the elementary texts and few users, from other areas especially, are consequently exposed to the deeper properties of the transforms. A little knowledge of analytic functions does go a long way, however, as will be illustrated in the examples.

2. PROPERTIES

A cheap, but useful, existence result for the transform results from the observation that the kernel is continuous on [0,L]. Consequently, if f is integrable on [0,L], $\hat{f}_L(s)$ exists. Not only is $\hat{f}_L(s)$ an entire function, but its derivatives satisfy the simple relation

$$D_s^k \hat{f}_L(s) = \int_0^L e^{-st}(-t)^k F(t)dt .$$

The transform of the derivative of a function readily follows from integration by parts.

We have

$$\mathcal{L}_L\{f'(t)\} = s\hat{f}_L(s) + e^{-sL}f(L-) - f(0+) ,$$

provided f is continuous on (0,L) and the interior limits at the endpoints exist. Whenever f has a discontinuity at the point a with $0 < a < L$, the expression $e^{-as}(f(a-)-f(a+))$ must be added, however.

Two useful forms for the transforms of integrals can also be obtained simply from integration by parts. We have

$$\mathcal{L}_L\left\{ \int_0^t f(\nu)d\nu \right\} = s^{-1}\hat{f}_L(s) - s^{-1}e^{-sL}\hat{f}_L(0) ,$$

$$\mathcal{L}_L\left\{ \int_0^L f(\nu)d\nu \right\} = -s^{-1}\hat{f}_L(s) + s^{-1}\hat{f}_L(0) ,$$

Iterations of the formulas for the transforms of the derivative or the integral directly produce the higher order properties.

The rescaling properties are not as useful in this finite case, since they alter the interval. The exponential shift property

$$\mathcal{L}_L\{ e^{-ct} f(t) \} = \hat{f}_L(s+c) ,$$

however, follows directly from the definition.

The convolution for the ordinary Laplace transformation for $0 < t < L$ uses only input from the functions for $0 < t < L$. Consequently, for that interval we have

$$\hat{f}_L(s)\hat{g}_L(s) = \mathcal{L}_L\left\{ \int_0^t f(t-\nu)g(\nu)d\nu \right\}$$

(The Laplace convolution for functions which satisfy $f(\nu) = g(\nu) = 0$ for $\nu < 0$ and for $\nu > L$ reduces to integration over the interval $[\max(t-L,0), \min(t,L)]$. However, for $0 < t < L$ this coincides with $[0,t]$.)

Some simple examples of transform pairs are

$$\mathcal{L}_L\{t\} = s^{-2} - e^{-Ls}(Ls^{-1} + s^{-2}) ,$$

$$\mathcal{L}_L\{e^{-ct}\} = (s+c)^{-1}\left(1 - e^{-L(s+c)}\right) ,$$

$$\mathcal{L}_L\{u(t-a)\} = s^{-1}(e^{-as} - e^{-Ls}) , \quad 0 < a < L .$$

$$\mathcal{L}_L\{\delta(t-a)\} = e^{-as} , \quad 0 < a < L.$$

Here δ represents the Dirac δ-function; the theory can be generalized in various ways so as to include it.

3. A BOUNDARY VALUE PROBLEM

A simple example for the illustration of the method is the boundary value problem

$$y''(x) + b^2y(x) = 0 , \quad y(0) = y(L) = 0, \quad b>0.$$

The transformed problem incorporates four boundary conditions, only two of which are given,

$$s^2\hat{y}_L(s) + (e^{-Ls}y'(L-) - y'(0+)) + b^2\hat{y}_L(s) = 0.$$

Somehow we must evaluate $y'(L-)$ and $y'(0+)$. The algebraic equation can be solved,

$$\hat{y}_L(s) = -\frac{e^{-Ls}y'(L-)-y'(0+)}{s^2+b^2} .$$

If the functions in the original problem are assumed to be bounded and integrable, then $\hat{y}_L(s)$ must be an entire function. Hence the numerator must be zero at $s = \pm ib$. The two conditions

$$e^{-ibL}y'(L-) - y'(0+) = 0$$

$$e^{ibL}y'(L-) - y'(0+) = 0$$

must be satisfied. We refer to such conditions as *compatibility conditions*. It is from such conditions that the extra boundary conditions can be studied. If $bL = k\pi$, then $y'(L-) = (-1)^k y'(0+)$ and there are an infinite number of solutions,

$$\hat{y}_L(s) = y'(0+)\frac{1-(-1)^k e^{-Ls}}{s^2+(k\pi/L)^2}$$

where $y'(0+)$ is arbitrary. The exponential shift for the ordinary Laplace transformation has the property

$$e^{-Ls}\hat{f}(s) = L\{f(t-L)u(t-L)\} \quad \text{for} \quad L > 0.$$

Consequently, terms multiplied by e^{-Ls} can be disregarded in the inversion, since we are interested only in the interval [0,L]. As a consequence, ordinary inversion tables, such as Roberts and Kaufman [10], can be used for the inversion of the remaining terms .

Hence

$$y(x) = y(0+)\frac{\sin(k\pi x/L)}{(k\pi/L)}$$

In the other case, if $bL \neq k\pi$, then the compatibility conditions cannot be satisfied, except by the trivial solution $y(L-) = y(0+) = 0$. Hence only the trivial solution $y(x) \equiv 0$ exists.

An initial value problem could be handled similarly. In fact, we could just as well consider the general linear homogeneous boundary conditions

$$\alpha_o y(0+) + \beta_o y'(0+) = 0, \qquad \alpha_o^2 + \beta_o^2 \neq 0\ ,$$

$$\alpha_L y(L-) + \beta_L y'(L-) = 0, \qquad \alpha_L^2 + \beta_L^2 \neq 0\ ,$$

These could be used to eliminate two of the four boundary values, then the compatibility conditions can be used to relate the remaining two values. As with the ordinary Laplace transformation, if the boundary conditions are non-homogeneous, the general method is not altered, merely the computational details are more complicated.If b = 0 the problem is only slightly altered by the appearance of a double root of the denominator of $\hat{y}_L(s)$. Hence at s = 0 both the numerator and its derivative must be set to zero to obtain the compatibility conditions

$$y'(L-) - y'(0+) = 0,$$

$$-Ly'(L-)-y'(0+) = 0.$$

These lead only to the trivial solution $y(x) \equiv 0$. All of the ideas of this paragraph carry over to the case of the non-homogeneous differential equation. Complications occur only in the details of the computations.

Were the differential equation non-homogeneous,

$$y''(x) + b^2y(x) = f(x), \quad b \neq 0,$$

the compatibility conditions would read,

$$e^{-ibL}\, y'(L-) - y'(0+) = \hat{f}_L(ib)\ ,$$

$$e^{ibL}\, y'(L-) - y'(0+) = \hat{f}_L(-ib)\ ,$$

which are non-homogeneous. Elimination of y'(L-) results in

$$(2i\sin(bL))y'(0+) = e^{-ibL}\hat{f}_L(-ib) - e^{ibL}\hat{f}_L(ib)\ .$$

Here we see that if $bL = k\pi$, then there is no solution for $y'(0+)$ unless the right hand side of the equation is also zero; that is, unless

$$\int_0^L \sin(b(L-\nu))\, f(\nu)d\nu = 0.$$

In that case $y'(0+)$ is arbitrary. Similarly, there is no solution for $y'(L-)$ unless

$$\int_0^L \sin(b\nu)\, f(\nu)d\nu = 0$$

On the other hand if $bL \neq k\pi$ there is a unique solution for the previously unassigned boundary values

$$y'(0+) = \int_0^L \frac{\sin(b(L-\nu))}{\sin(bL)} f(\nu)d\nu\ ,$$

$$y'(L-) = \int_0^L \frac{\sin(b\nu)}{\sin(bL)} f(\nu)d\nu\ ,$$

Inversion is now easy, we ignore the terms which have the factor e^{-Ls} and we use the ordinary Laplace convolution to obtain, for $bL \neq k\pi$,

$$y(x) = \int_0^x b^{-1}\sin(b(x-\nu))f(\nu)d\nu) +$$

$$+\ b^{-1}\sin(bx)\int_0^L \frac{\sin(b(L-\nu))}{\sin(bL)} f(\nu)d\nu\ , \qquad 0 < x < L.$$

4. A PARTIAL DIFFERENTIAL EQUATION PROBLEM

A more complicated example is provided by the one dimensional heat equation with a source term

$$u_t(x,t) = u_{xx}(x,t) + \phi(x,t)\ , \qquad t>0, \quad 0<x<L.$$

We assume an initial temperature profile of $f(x)$, insulation at $x = L$, and that the temperature at $x = 0$ is described by $g(t)$. An expression for the temperature at $x = L$ is desired; that is, we want a formula for $u(L,t)$. Thus the initial and boundary conditions can be written

$$u(x,0) = f(x), \qquad 0<x<L,$$

$$u_x(L,t) = 0 \;, \qquad t>0,$$

$$u(0,t) = g(t) \;, \qquad t>0$$

The application of the finite Laplace transformation converts the problem into

$$\hat{u}_t(z,t) = z^2\hat{u}(z,t) + z(e^{-Lz}u(L,t)-g(t)) - u_x(0,t) + \hat{\phi}(z,t),$$

$$\hat{u}(z,0) = \hat{f}(z).$$

It is convenient next to apply the Laplace transformation so that an algebraic equation is obtained. This equation can be solved for

$$\hat{\hat{u}}(z,s) = \frac{\hat{f}(z) + ze^{-Lz}\hat{u}(L,s) - z\hat{g}(z) - u_x(0,s) + \hat{\hat{\phi}}(z,s)}{s-z^2}$$

Since $\hat{\hat{u}}(z,s)$ must be an entire function of z, the numerator must be zero at $z=\pm s^{1/2}$. This produces two compatibility conditions

$$\hat{f}(s^{1/2}) + s^{1/2}e^{Ls^{1/2}}\hat{u}(L,s) - s^{1/2}\hat{g}(s) - \hat{u}(s) - \hat{u}_x(0,s)$$

$$+ \hat{\hat{\phi}}(s^{1/2},s) = 0 \;,$$

$$\hat{f}(-s^{1/2}) - s^{1/2}e^{-Ls^{1/2}}\hat{u}(L,s) + s^{1/2}\hat{g}(s) - \hat{u}_x(0,s) +$$

$$+ \hat{\hat{\phi}}(-s^{1/2},s) = 0.$$

These equations can be solved for the transforms of the unknown boundary conditions; we have

$$\hat{u}(L,s) = \frac{-\hat{f}(s^{1/2}) + \hat{f}(-s^{1/2}) + 2s^{1/2}\hat{g}(s) - \hat{\hat{\phi}}(s^{1/2}) + \hat{\hat{\phi}}(-s^{1/2},s)}{2\cosh(Ls^{1/2})}$$

which yet needs to be inverted.

In order to carry out the inversion some formulas which involve heat kernels are required. From Roberts and Kaufman [10]

$$L^{-1}\left\{e^{-Ls^{1/2}}\right\} = \psi(L,t) = L(4\pi t^3)^{-1/2}\exp(-L/(4t)) \;,$$

$$L^{-1}\left\{s^{-1/2}e^{-Ls^{1/2}}\right\} = \chi(L,t) = (\pi t)^{-1/2}\exp(-L/(4t)) \;,$$

$$L^{-1}\left\{(1/2)\,\mathrm{sech}\,(Ls^{1/2})\right\} = -(2L)^{-2}\,\partial\theta_1(\nu/2\,|\,t/L^2)/\partial\nu\big|_{\nu=0}$$

$$= 2^{-2}\sum_{n=-\infty}^{+\infty}(-1)^n\psi((n-1/2)L,t) = k(t).$$

For the inversion of $\hat{f}(s^{1/2})$ we need and *Efros formula* for the finite transformation. Various cases of such formulas appear in the tables [10]; the first general result is due to Efros [6]. Assuming absolutely convergent integrals, the desired formula can be derived directly from the definition of the tansform as follows.

$$\hat{f}(s^{1/2}) = \int_0^L e^{-ws^{1/2}} f(w)dw$$

$$= \int_0^L \left(\int_0^\infty e^{-st}\psi(w,t)dt\right) f(w)dw$$

$$= L\left\{\int_0^L \psi(w,t)f(w)dw\right\}$$

The result for $\hat{f}(-s^{1/2})$ is an analog; hence we have

$$L^{-1}\left\{-\hat{f}(s^{1/2}) + \hat{f}(-s^{1/2})\right\} = \int_0^L (\psi(-w,t) - \psi(w,t))f(w)dw.$$

In a similar manner we obtain

$$\hat{\hat{\phi}}(s^{1/2},s) = \int_0^L\int_0^\infty e^{-ws^{1/2}-\nu s}\phi(w,\nu)d\nu dw$$

$$= \int_0^L\int_0^\infty\int_0^\infty e^{-sx-s\nu}\psi(w,x)\phi(w,\nu)dx d\nu dw$$

$$= \int_0^\infty e^{-st}\int_0^L\int_0^t \psi(w,t-\nu)\phi(w,\nu)d\nu dw dx$$

$$= L\left\{\int_0^L\int_0^t \psi(w,t-\nu)\phi(w,\nu)d\nu dw\right\}.$$

Consequently,

$$L^{-1}\left\{ -\hat{\hat{\phi}}(s^{1/2},s) + \hat{\hat{\phi}}(-s^{1/2},s) \right\} =$$

$$= \int_o^L \int_o^t (\psi(-w,t-\nu) - \psi(w,t-\nu))\phi(w,\nu)d\nu dw \ .$$

We can now write u(L,t) in terms of convolutions with the function k(t),

$$u(L,t) = \left(\int_o^L (\psi(-w,t)-\psi(w,t))f(w)dw\right) * k(t)$$

$$+ D_t(g(t) * \chi(0,t) * k(t))$$

$$+ \left(\int_o^L \int_o^t (\psi(-w,t-\nu) - \psi(w,t-\nu))\ \phi(w,\nu)dwd\nu\right) * k(t).$$

Because the heat kernels satisfy the convolution identities,

$$\chi(a,t) * \psi(b,t) = \chi(a+b,t)\ ,$$

$$\psi(a,t) * \psi(b,t) = \psi(a+b,t),$$

which are easy to obtain from the Laplace transforms, we can rewrite the convolutions.

Hence our temperature result can be expressed as

$$u(L,t) =$$

$$= 2^{-2} \sum_{n=-\infty}^{+\infty} (-1)^n \int_o^L (\psi((n-1/2)L - w,t) -$$

$$\psi((n-1/2)L + w,t)))\ f(w)dw$$

$$+ 2^{-1} \sum_{n=-\infty}^{+\infty} D_t \int_o^t \chi((n-1/2)L,t-\nu)g(\nu)d\nu$$

$$= 2^{-2} \sum_{n=-\infty}^{+\infty} (-1)^n \int_o^L \int_o^t (\psi((n-1/2)L - w,t-\nu) -$$

$$-\psi((n-1/2)L + w,t-\nu))\phi(w,\nu)d\nu dw.$$

Several points should be noted. First, both of the heat kernels decay rapidly with $|n|$; in fact, they behave something like $\exp(-cn^2)$. Second, the Efros formulas for the finite interval correspond to those for the infinite interval, suitably chopped. Third, we did not need to compute the full solution u(x,t) for arbitrary x, and then evaluate u(L,t). Fourth, the flux at x = 0 can similarly be obtained. Fifth, u(x,t) could also be obtained, we simply use of the compatibility conditions to eliminate $\hat{u}(L,s)$ and $\hat{u}_x(0,s)$, and then we invert that result.

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