Rev. Téc. Ing., Univ. Zulia
Vol. 11, No. 1, 1988

SUMMATION OF CERTAIN INFINITE SERIES INVOLVING

$$\delta_n = \frac{1.3.5\ \ldots\ (2n-1)}{2.4.6\ \ldots\ (2n)} \quad (*)$$

W.A. Bassali
Department of Mathematics,
Faculty of Science
Kuwait University
13060 Kuwait

ABSTRACT

The object of the present paper is to find the sums of the following infinite series:

$$\sum_{n=1}^{\infty} \delta_{n+m} x^{n-1}, \quad \sum_{n=1}^{\infty} \frac{\delta_{n+m}}{n} x^n, \quad \sum_{n=1}^{\infty} \frac{\delta_{n+m}}{2n-1} x^n,$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n+m} \delta_n, \quad \sum_{n=0}^{\infty} \frac{\delta_n}{(n+m+1)^\lambda}, \quad \sum_{0}^{\infty} \frac{\delta_n}{(2n+2m+1)^\lambda}$$

$$(\lambda = 1,2,3), \quad \sum_{n=0}^{\infty} \frac{\delta_n}{(2n-1)(n+m+1)^2},$$

$$\sum_{n=0}^{\infty} \frac{\delta_n}{(2n-1)(2n+2m+1)^2}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n \delta_n}{(n+m+1)^2},$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \delta_n}{(2n+2m+1)^2},$$

where $m = 0,1,2,\ldots$, $-1 \le x \le 1$. Some particular cases are mentioned.
Simpler proofs of the Shafer-Knuth formula are given.

RESUMEN

El objeto del presente trabajo es evaluar las siguientes series infinitas:

$$\sum_{n=1}^{\infty} \delta_{n+m} x^{n-1}, \quad \sum_{n=1}^{\infty} \frac{\delta_{n+m}}{n} x^n,$$

$$\sum_{n=1}^{\infty} \frac{\delta_{n+m}}{2n-1} x^n, \quad \sum_{n=0}^{\infty} \frac{x^n}{n+m} \delta_n,$$

$$\sum_{n=0}^{\infty} \frac{\delta_n}{(n+m+1)^\lambda}, \quad \sum_{0}^{\infty} \frac{\delta_n}{(2n+2m+1)^\lambda} \quad (\lambda = 1,2,3),$$

$$\sum_{n=0}^{\infty} \frac{\delta_n}{(2n-1)(n+m+1)^2}, \quad \sum_{n=0}^{\infty} \frac{\delta_n}{(2n-1)(2n+2m+1)^2},$$

$$\sum_{0}^{\infty} \frac{(-1)^n \delta_n}{(n+m+1)^2}, \quad \sum_{0}^{\infty} \frac{(-1)^n \delta_n}{(2n+2m+1)^2},$$

donde $m = 0,1,2,\ldots$, $-1 \le x \le 1$. Se menciona algunos casos particulares. Se dan demostraciones sencillas a la fórmula de Shafer-Knuth.

1. INTRODUCTION

An expository article dealing with the result

$$\sum_{1}^{\infty} \frac{\delta_n}{n} = \ln 4 \tag{1}$$

was given by Ross [9]. The same result was obtained in connection with a probability problem by Callan [2]. The two formulae mentioned at the end of Callan's paper are equivalent to

$$\sum_{1}^{\infty} \delta_n x^{n-1} = \frac{1}{1-x+\sqrt{1-x}} \quad (0 \le x < 1), \tag{2}$$

$$\sum_{1}^{\infty} \frac{\delta_n}{n} x^n = 2 \ln \frac{2}{1+\sqrt{1-x}} \quad (-1 \le x \le 1). \tag{3}$$

(*) $\delta_n = \dfrac{(2n)!}{2^{2n}(n!)^2}$

The proof of (2) follows directly from the generating function

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \delta_n x^n (\delta_0 = 1, -1 \leq x < 1) \quad , \tag{4}$$

while (3) is obtained by integrating both sides of (2) from 0 to x.

The extension

$$\sum_{n=1}^{\infty} \frac{\delta_{n+m}}{n} = \delta_m \left[\ln 4 + \sum_{n=1}^{m} \frac{1}{n} \right] \tag{5}$$

of (1) was suggested by Shafer [10] and proved by Knuth [7]. In a recent paper by Lehmer [8], several interesting infinite series of the types

$$\sum_{n=0}^{\infty} a_n \delta_n \quad \text{and} \quad \sum_{n=0}^{\infty} a_n / \delta_n \quad \text{are summed, the}$$

a_n's being very simple functions of n.

In this paper various generalisations of the foregoing results are established and simpler proofs of (5) are given.

2. EXTENSIONS OF CALLAN'S FORMULAE

In this section we prove the following generalisations of (2), (3) and (5):

$$\sum_{n=1}^{\infty} \delta_{n+m} x^{n-1} = u^{-m-1} \sum_{n=0}^{m} \alpha_n (1-x)^{(n-1)/2} (-1 \leq x < 1), \tag{6}$$

$$\sum_{1}^{\infty} \frac{\delta_{n+m}}{n} x^n = 2\left[\beta_m \ln \frac{2}{u} + \sum_{n=1}^{m} \frac{\beta_{m-n}}{n} (u^{-n} - 2^{-n})\right] (-1 \leq x \leq 1), \tag{7}$$

where m is a positive integer,

$$u = 1 + \sqrt{1-x} \quad , \tag{8}$$

and the α_n's and β_n's are constants to be determined.

Proof. From (4) we see that

$$\sum_{n=1}^{\infty} \delta_{n+m} x^{n-1} = x^{-m-1} \left[\frac{1}{\sqrt{1-x}} - \sum_{n=0}^{m} \delta_n x^n \right] . \tag{9}$$

Setting $x = 1-t^2$ in the right side of (9) we find that it takes the form

$$\frac{1}{t(1-t^2)^{m+1}} \left[1 - t \sum_{n=0}^{m} \delta_n (1-t^2)^n \right] \tag{10}$$

The expression between the square brackets in (10) is a polynomial of degree 2m+1 in t. By mathematical induction it can be proved that this polynomial is divisible by $(1-t)^{m+1}$ and (10) may be written as

$$(1+t)^{-m-1} \sum_{n=0}^{m} \alpha_n t^{n-1} = u^{-m-1} \sum_{n=0}^{m} \alpha_n (1-x)^{(n-1)/2} , \tag{11}$$

where the α_n's are real constants that satisfy the following recurrence relations:

$$\alpha_{2n} = \sum_{k=1}^{2n} (-1)^{k+1} \binom{m+1}{k} \alpha_{2n-k} \quad (2n \leq m+1), \tag{12a}$$

$$\alpha_{2n+1} = \sum_{k=1}^{2n+1} (-1)^{k+1} \binom{m+1}{k} \alpha_{2n+1-k} + (-1)^{n+1} \sum_{k=n}^{m} \binom{k}{n} \delta_k , \tag{12b}$$

$$\alpha_{m-2n} = \sum_{k=0}^{2n-1} (-1)^{k+1} \binom{m+1}{2n-k} \alpha_{m-k} + (-1)^n \sum_{k=0}^{n} \binom{m-k}{n-k} \delta_{m-k} , \tag{13a}$$

$$\alpha_{m-2n-1} = \sum_{k=0}^{2n} (-1)^k \binom{m+1}{2n+1-k} \alpha_{m-k} \tag{13b}$$

Equations (12a,b) express α_k in terms of $\alpha_0 = 1$, α_1, $\alpha_2, \ldots, \alpha_{k-1}$ while (13a,b) give α_k in terms of $\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_m = \delta_m$. It is easily seen that

$$\alpha_0 = 1, \ \alpha_1 = m - \sum_{1}^{m} \delta_\lambda, \ \alpha_2 = (m+1)\left[\frac{1}{2} m - \sum_{1}^{m} \delta_n\right] , \ \ldots \tag{14a}$$

$$\alpha_m = \delta_m,\ \alpha_{m-1} = (m+1)\,\delta_m,\ \alpha_{m-2} =$$

$$= \frac{1}{2}\,(m^2 + m + 2)\,\delta_m - \delta_{m-1},\ \ldots \qquad (14b)$$

$$\sum_0^m \alpha_n = 2^{m+1}\,\delta_{m+1}\ ,\ \sum_0^m (-1)^n\,\alpha_n = 2^{-m}\ . \qquad (14c)$$

Values of α_n ($n= 0,1,2,\ldots,m$) corresponding to $m= 1,2,\ldots,6$ are presented in Table 1.

Table 1

m	α_n $(n = 0,1,2,\ldots,m)$						
1	1 ,	$\frac{1}{2}$					
2	1 ,	$\frac{9}{8}$,	$\frac{3}{8}$				
3	1 ,	$\frac{29}{16}$,	$\frac{5}{4}$,	$\frac{5}{16}$			
4	1 ,	$\frac{325}{128}$,	$\frac{345}{128}$,	$\frac{175}{128}$,	$\frac{35}{128}$		
5	1 ,	$\frac{843}{256}$,	$\frac{609}{128}$,	$\frac{469}{128}$,	$\frac{189}{128}$,	$\frac{63}{256}$	
6	1 ,	$\frac{4167}{1024}$,	$\frac{7651}{1024}$,	$\frac{3969}{512}$,	$\frac{2415}{512}$,	$\frac{1617}{1024}$,	$\frac{231}{1024}$

For $m = 0$ the formula (6) reduces to (2). Multiplying both sides of (6) by dx and integrating from 0 to x we get (7) in which the β_n's are defined by

$$\sum_{n=0}^{m} \beta_n\, t^n = \sum_{n=0}^{m} \alpha_n\,(t-1)^n\ , \qquad (15)$$

so that

$$\beta_n = \sum_{\nu=n}^{m} (-1)^{\nu-n} \binom{\nu}{n}\,\alpha_\nu \quad (n= 0,1,2,\ldots,m), \qquad (16)$$

$$\beta_0 = 2^{-m}\ ,\ \beta_m = \beta_{m-1} = \delta_m\ ,\ \sum_{n=0}^{m} 2^n \beta_n = 2^{m+1}\,\delta_{m+1}\,. \qquad (17)$$

Setting $x = 1$ in (7) and comparing the result with (5) we obtain

$$\sum_{n=1}^{m} \frac{1-2^{-n}}{n}\,\beta_{m-n} = \frac{1}{2}\,\beta_m \sum_{n=1}^{m} \frac{1}{n} \qquad (18)$$

Table 2 contains values of β_n ($n= 0,1,2,\ldots,m$) corresponding to $m= 1,2,\ldots,6$.

Table 2

m	β_n $(n = 0,1,2,\ldots,m)$						
1	$\frac{1}{2}$,	$\frac{1}{2}$					
2	$\frac{1}{4}$,	$\frac{3}{8}$,	$\frac{3}{8}$				
3	$\frac{1}{8}$,	$\frac{1}{4}$,	$\frac{5}{16}$,	$\frac{5}{16}$			
4	$\frac{1}{16}$,	$\frac{5}{32}$,	$\frac{15}{64}$,	$\frac{35}{128}$,	$\frac{35}{128}$		
5	$\frac{1}{32}$,	$\frac{3}{32}$,	$\frac{21}{128}$,	$\frac{7}{32}$,	$\frac{63}{256}$,	$\frac{63}{256}$	
6	$\frac{1}{64}$,	$\frac{7}{128}$,	$\frac{7}{64}$,	$\frac{21}{128}$,	$\frac{105}{512}$,	$\frac{231}{1024}$,	$\frac{231}{1024}$

For $m= 0$ (7) reduces to (3). For $m= 1,2,3$ and $x= -1$ we get

$$\sum_1^\infty \frac{(-1)^{n-1}\,\delta_{n+1}}{n} = \ln\frac{\sqrt{2}+1}{2} + \frac{3}{2} - \sqrt{2}\ , \qquad (19a)$$

$$\sum_1^\infty \frac{(-1)^{n-1}\,\delta_{n+2}}{n} = \frac{1}{4}\left[3\ln\frac{\sqrt{2}+1}{2} + \frac{7}{4} - \sqrt{2}\,\right], \qquad (19b)$$

$$\sum_1^\infty \frac{(-1)^{n-1}\,\delta_{n+3}}{n} = \frac{1}{8}\left[5\ln\frac{\sqrt{2}+1}{2} + \frac{27}{4} - \frac{13\sqrt{2}}{3}\,\right]. \qquad (19c)$$

3. PROOFS OF THE SHAFER-KNUTH FORMULA

The formula 4.387, 6.p.588 of [3] may be written as

$$\int_0^{\pi/2} \cos^{2m} x\,\ln\sin x\,dx = -\frac{\pi}{4}\,\delta_m\left(\ln 4 + \sum_{n=1}^{m}\frac{1}{n}\right). \qquad (20)$$

Setting $\cos x = t$ in the left side gives the definite integral

$$I = \frac{1}{2}\int_0^1 \frac{t^{2m}\ln(1-t^2)\,dt}{\sqrt{1-t^2}}$$

Expanding the logarithmic function in powers of t and putting t= sinθ we find

$$I=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\pi/2}\sin^{2n+2m}\theta\,d\theta$$

Applying the formula

$$\int_0^{\pi/2}\sin^{2k}\theta\,d\theta=\frac{\pi}{2}\delta_k \quad (k=0,1,2,\ldots),$$

we obtain (5). It is to be noted that the formula (5) is a special case of the general functional relation

$$\psi(\lambda)-\psi(\lambda-\nu)=\frac{\Gamma(\lambda)}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)} \quad (\text{Re }\lambda>\text{Re }\nu\geq 0),$$

given by Kalla and Ross [5]. Setting $\lambda = m+1$, $\nu= m+\frac{1}{2}$ and noting that

$$\delta_m=\Gamma(m+\tfrac{1}{2})/\sqrt{\pi}\,\Gamma(m+1),\quad \psi(m+1)=-\gamma+\sum_{n=1}^{m}n^{-1},$$

$$\psi(\tfrac{1}{2})=-\gamma-\ln 4,$$

where γ is Euler's constant we obtain (5). A more general functional relation is established by Kalla and Bader Al-Saqabi [6].

4. SUM OF THE INFINITE SERIES

$$S_m(x)=\sum_{0}^{\infty}\frac{x^n}{n+m}\,\delta_n \quad (m>0,\ -1\leq x\leq 1). \tag{21}$$

If $0<x\leq 1$ we have

$$S_m(x)=\frac{2}{x^m}\int_{\sqrt{1-x}}^{1}(1-t^2)^{m-1}\,dt=\frac{2}{x^m}\int_0^{\sin^{-1}\sqrt{x}}\sin^{2m-1}\theta\,d\theta, \tag{22}$$

$$S_m(1)=\sqrt{\pi}\,\Gamma(m)/\Gamma(m+\tfrac{1}{2}). \tag{23}$$

This formula agrees with the series (1009), p. 186 of [4]. Equations (1) and (23) show that

$$\lim_{m\to 0}\left[\frac{\sqrt{\pi}\,\Gamma(m)}{\Gamma(m+\frac{1}{2})}-\frac{1}{m}\right]=\ln 4. \tag{24}$$

we also have

$$S_m(-x)=\frac{2}{x^m}\int_1^{\sqrt{1+x}}(t^2-1)^{m-1}\,dt=\frac{2}{x^m}\int_0^{\tan^{-1}\sqrt{x}}\tan^{2m-1}\theta\,\sec\theta\,d\theta. \tag{25}$$

If m is a positive integer then

$$S_m(x)=\frac{2}{x^m}\sum_{\nu=0}^{m-1}\frac{(-1)^\nu}{2\nu+1}\binom{m-1}{\nu}\{1-(1-x)^{\nu+1/2}\}\ (-1\leq x\leq 1), \tag{26}$$

$$S_m(1)=2\sum_{\nu=0}^{m-1}\frac{(-1)^\nu}{2\nu+1}\binom{m-1}{\nu}=\frac{2}{(2m-1)\,\delta_{m-1}}=\frac{1}{m\delta_m}; \tag{27}$$

$$S_1(1)=2,\ S_2(1)=\frac{4}{3},\ S_3(1)=\frac{16}{15}, \tag{28}$$

$$S_1(-1)=2\sqrt{2}-2,\ S_2(-1)=\frac{2}{3}(2-\sqrt{2}),\ S_3(-1)==\frac{2}{15}(7\sqrt{2}-8). \tag{29}$$

From (3) and (26) we get

$$\sum_1^{\infty}\frac{x^{n+m}\delta_n}{n(n+m)}=\frac{x^m}{m}\left(\frac{1}{m}+2\ln\frac{2}{u}\right)-\frac{2}{m}\sum_{\nu=0}^{m-1}\frac{(-1)^\nu}{2\nu+1}\binom{m-1}{\nu}\{1-(1-x)^{\nu+1/2}\}, \tag{30}$$

where u is defined by (7). For m=1 this reduces to

$$\sum_1^{\infty}\frac{x^{n+1}\delta_n}{n(n+1)}=2x\ln\frac{2}{u}+x-2+2\sqrt{1-x}\quad (|x|\leq 1), \tag{31}$$

which does not agree with equation (7), p.451 of [8] due to misprint*.

If $m = k + \frac{1}{2}$, where k is a positive integer then

$$S_{k+1/2}(-1) = 2\sum_{0}^{\infty} \frac{(-1)^n \delta_n}{2n+2k+1} = 2\int_0^{\pi/4} \sec\theta(\sec^2\theta - 1)^k \, d\theta. \tag{32}$$

From this we have

$$S_{1/2}(-1) = 2\ln(\sqrt{2}+1), \quad S_{3/2}(-1) = \sqrt{2} - \ln(\sqrt{2}+1),$$

$$\left.\begin{aligned} &S_{5/2}(-1) = \frac{3}{4}\ln(\sqrt{2}+1) - \frac{\sqrt{2}}{4}, \quad S_{7/2}(-1) = \\ &= \frac{13}{24}\sqrt{2} - \frac{5}{8}\ln(\sqrt{2}+1) . \end{aligned}\right\} \tag{33}$$

5. SUM OF THE INFINITE SERIES

$$I_m(z) = \sum_{1}^{\infty} \frac{\delta_{n+m}}{2n-1} z^n \quad (m = 0,1,2,\ldots,\ |z| \le 1). \tag{34}$$

Following a procedure similar to that adopted in Section 2 we find

$$I_m(z) = 2^{-m}\sqrt{z}\sum_{n=0}^{m} 2^n \beta_n \int_0^{(1-\sqrt{1-z})/\sqrt{z}} (1+t^2)^{m-n} dt,$$

$$I_m(1) = 2^{-m}\sum_{n=0}^{m} 2^n \beta_n \int_0^1 (1+t^2)^{m-n} dt, \tag{36}$$

where the β_n's corresponding to values of m are given in Table 2. Thus we get the following results:

$$I_0(z) = 1 - \sqrt{1-z}, \quad I_0(1) = 1, \quad I_0(-1) = 1 - \sqrt{2}\ ; \tag{37}$$

$$I_1(z) = \frac{1}{6z}(1 - \sqrt{1-z})(1 + 4z - \sqrt{1-z}),$$

$$I_1(1) = 5/6, \quad I_1(-1) = (1 - 2\sqrt{2})/6\ ; \tag{38}$$

$$I_2(z) = \frac{1}{120 z^2}(1 - \sqrt{1-z})\,[12 + 13z + 64z^2 -$$

$$- (12+19z)\sqrt{1-z}\,],$$

$$I_2(1) = 89/120, \quad I_2(-1) = (49 - 56\sqrt{2})/120\ ; \tag{39}$$

$$I_3(z) = \frac{1-\sqrt{1-z}}{560 z^3}[40 + 38z + 47z^2 + 256z^3 -$$

$$- (40+58z+81z^2)\sqrt{1-z}\,],$$

$$I_3(1) = 381/560, \quad I_3(-1) = 9(9 - 16\sqrt{2})/560 \tag{40}$$

6. SUMS OF THE INFINITE SERIES

$$J_m = \sum_{n=0}^{\infty} \frac{\delta_n}{(n+m+1)^2}, \quad J'_m = \sum_{0}^{\infty} \frac{\delta_n}{(2n+2m+1)^2}$$

$$(m = 0,1,2,\ldots) \tag{41}$$

Consider the definite integral

$$L_m = \int_0^1 \frac{t^{2m+1}\ln t}{\sqrt{1-t^2}} dt . \tag{42}$$

Using the generating function (4) and applying the formula

$$\int_0^1 t^k \ln t \, dt = -(k+1)^{-2} \quad (k > -1),$$

we see that $L_m = -\frac{1}{4} J_m$. Setting $t = \sin\theta$ in (42) yields

$$L_m = \int_0^{\pi/2} \sin^{2m+1}\theta \ln\sin\theta \, d\theta =$$

$$= \frac{1}{(2m+1)\delta_m}\left[\ln 2 + \sum_{k=1}^{2m+1} \frac{(-1)^k}{k}\right], \tag{43}$$

on using the formula 4.387, 5.p.588 of [3]. Writing (42) in the form

$$L_m = \frac{1}{4}\int_0^1 \frac{(1-u)^m \ln(1-u)}{\sqrt{u}} du ,$$

---

* the term + x(ln 4-1) in this equation must be replaced by -x(ln 4-1).

expanding the logarithmic function as a power series in u and integrating term by term we arrive at the expression

$$-2^{m-1}\, m! \sum_{n=1}^{\infty} \frac{(2n-1)!!}{n(2n+2m+1)!!}\ .$$

Collecting these results we have

$$J_m = -4 \int_0^{\pi/2} \sin^{2m+1}\theta \,\ln \sin\theta \, d\theta$$

$$= \frac{-4}{(2m+1)\delta_m} [\ln 2 + \sum_{k=1}^{2m+1} \frac{(-1)^k}{k}] =$$

$$= 2^{m+1}\, m! \sum_{n=1}^{\infty} \frac{(2n-1)!!}{n(2n+2m+1)!!}\ . \qquad (44)$$

Similar considerations lead to

$$J'_m = -\int_0^1 \frac{t^{2m}\ln t}{\sqrt{1-t^2}}\, dt = -\int_0^{\pi/2} \sin^{2m}\theta \,\ln \sin\theta \, d\theta$$

$$= \frac{\pi}{2}\delta_m [\ln 2 + \sum_{k=1}^{2m} \frac{(-1)^k}{k}] =$$

$$= \frac{\pi(2m-1)!!}{2^{m-2}} \sum_{n=1}^{\infty} \frac{(n-1)!\,\delta_n}{(n+m)!}\ , \qquad (45)$$

where the formula 4.387, 4. p.588 of [3] is used. From (44) and (45) we get

$$J_0 = 4(1-\ln 2)\ ,\ J_1 = \frac{4}{9}(5-6\ln 2),$$

$$J_2 = \frac{8}{225}(47 - 60 \ln 2)\ ,$$

$$J_3 = \frac{16}{3675}(319 - 420 \ln 2)\ ,$$

$$J_4 = \frac{64}{99225}(1879 - 2520 \ln 2)\ ; \qquad (46)$$

$$J'_0 = \frac{\pi}{2}\ln 2,\ J'_1 = \frac{\pi}{8}(2\ln 2 - 1),\ J'_2 = \frac{\pi}{64}(12\ln 2 - 7),$$

$$J'_3 = \frac{\pi}{384}(60\ln 2 - 37),\ J'_4 = \frac{\pi}{6144}(840 \ln 2 - 533). \qquad (47)$$

It is worthy of mentioning here that

$$\sum_1^{\infty} \frac{\delta_n}{n^2} = \int_0^1 (1 - \frac{1}{\sqrt{1-x}}) \frac{\ln x}{x}\, dx =$$

$$= -4 \int_0^{\pi/2} \tan\frac{\theta}{2} \ln \sin\theta \, d\theta\ . \qquad (48)$$

Applying the two formulae 4.241, 3, 4, p.535 of [3] and using the expansion

$$\sqrt{1-x^2} = -\sum_0^{\infty} \frac{\delta_n x^{2n}}{2n-1}$$

we obtain

$$\sum_{n=0}^{\infty} \frac{\delta_n}{(2n-1)(n+m+1)^2} =$$

$$= \frac{4}{(2m+1(2m+3)\delta_m} [\sum_1^{2m+1} \frac{(-1)^k}{k} - \frac{1}{2m+3} + \ln 2]\ , \qquad (49a)$$

$$\sum_{n=0}^{\infty} \frac{\delta_n}{(2n-1)(2n+2m+1)^2} =$$

$$= \frac{\pi\delta_m}{4(m+1)} [\sum_1^{2m} \frac{(-1)^{k-1}}{k} - \frac{1}{2m+2} - \ln 2]\ . \qquad (49b)$$

7. SUMS OF THE INFINITE SERIES

$$K_m = \sum_0^{\infty} \frac{(-1)^n\delta_n}{(n+m+1)^2}\ ,\ K'_m = \sum_0^{\infty} \frac{(-1)^n\delta_n}{(2n+2m+1)^2}$$

$$(m = 0,1,2,\ldots) \qquad (50)$$

It is easily shown that

$$K_m = -4\int_0^1 \frac{t^{2m+1}\ln t}{\sqrt{1+t^2}}\, dt =$$

$$= -4\int_0^{\pi/4} \tan^{2m+1}\theta \sec\theta \ln\tan\theta \, d\theta$$

$$= 4(-1)^{m+1} \lim_{t\to 0} \int_t^{\pi/4} \ln \tan\theta \, d$$

$$\{ \sum_{\nu=0}^{m} \frac{(-1)^{\nu}}{2\nu+1} \binom{m}{\nu} \sec^{2\nu+1}\theta \}$$

$$= 4(-1)^{m} \lim_{t\to 0} \sum_{\nu=0}^{m} \frac{(-1)^{\nu}}{2\nu+1} \binom{m}{\nu}$$

$$\{\sec^{2\nu+1} \ln \tan t + \int_t^{\pi/4} \left[\frac{\sec^{2\nu+3}\theta - \sec\theta}{\tan\theta} + \operatorname{cosec}\theta\right] d\theta\}$$

$$= 4(-1)^{m} \lim_{t\to 0} \sum_{\nu=0}^{m} \frac{(-1)^{\nu}}{2\nu+1} \binom{m}{\nu} \{\sec^{2\nu+1} t \ln \tan t - \ln \tan \frac{t}{2} + \ln \tan \frac{\pi}{8} + \int_0^{\pi/4} \sum_{\lambda=0}^{\nu} \sec^{2\lambda}\theta \, d(\sec\theta)\}.$$

Applying the combinatorial identity (27) we obtain the closed expression

$$K_m = 4(-1)^m \left[\frac{\ln(2\sqrt{2}-2)}{(2m+1)\delta_m} + \sum_{\nu=0}^{m} \{ \frac{(-1)^{\nu}}{2\nu+1} \binom{m}{\nu} \sum_{\lambda=0}^{\nu} \frac{2^{\lambda}\sqrt{2} - 1}{2\lambda+1} \} \right]. \qquad (51)$$

Thus we have

$$K_0 = 4(\sqrt{2} - 1) + 4 \ln (2\sqrt{2} - 2), \qquad (52)$$

$$K_1 = \frac{4}{9} \left[5 - 4\sqrt{2} - 6 \ln (2\sqrt{2} - 2)\right], \qquad (53)$$

$$K_2 = \frac{8}{225} \left[60 \ln (2\sqrt{2} - 2) + 43\sqrt{2} - 47\right]. \qquad (54)$$

Similarly we have

$$K'_m = - \int_0^1 \frac{t^{2m} \ln t}{\sqrt{1+t^2}} dt = - \int_0^{\pi/4} \tan^{2m}\theta \sec\theta \ln \tan\theta \, d\theta, \qquad (55)$$

but no closed expression is available for this integral. For m = 0 we have

$$K'_0 = \sum_{0}^{\infty} \frac{(-1)^n \delta_n}{(2n+1)^2} = - \int_0^1 \frac{\ln t \, dt}{\sqrt{1+t^2}} =$$

$$- \int_0^{\pi/4} \sec\theta \ln \tan\theta \, d\theta = 0.955202 \qquad (56)$$

It is shown by Bassali [1] that any of the three following integrals have the same value:

$$\int_0^1 \frac{\sinh^{-1} x}{x} dx = \int_0^{\pi/2} \sin^{-1} \sin^2\theta \, d\theta =$$

$$\int_0^{\pi/2} \cos^{-1} \sqrt{\cos\theta} \, d\theta . \qquad (57)$$

These are to be compared with

$$\int_0^1 \frac{\tan^{-1} x}{x} dx = \frac{1}{2} \int_0^{\pi/2} \frac{\theta}{\sin\theta} d\theta = \sum_{0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

$$= \text{Catalan's constant } G = 0.915966 \qquad (58)$$

See series (990), p.182 and series (995), p. 184 of [4].

From (41) and (50) we deduce that

$$P_m = \sum_{n=0}^{\infty} \frac{\delta_{2n}}{(2n+m+1)^2} = \frac{1}{2} (J_m + K_m), \qquad (59)$$

$$Q_m = \sum_{n=1}^{\infty} \frac{\delta_{2n-1}}{(2n+m)^2} = \frac{1}{2} (J_m - K_m). \qquad (60)$$

Applying (46) and (52)-(54) we have

$$P_0 = 2\sqrt{2} + 2 \ln (\sqrt{2} - 1), \quad P_1 =$$

$$= \frac{4}{9} \left[5 - 2\sqrt{2} - 3 \ln (4\sqrt{2} - 4)\right],$$

$$P_2 = \frac{4}{225} \left[43\sqrt{2} + 60 \ln(\sqrt{2} - 1)\right]; \qquad (61)$$

$$Q_0 = 4 - 2\sqrt{2} - 2 \ln (4\sqrt{2} - 4),$$

$$Q_1 = \frac{4}{9} \left[2\sqrt{2} + 3 \ln(\sqrt{2} - 1)\right],$$

$$Q_2 = \frac{4}{225} \left[94 - 43\sqrt{2} - 60 \ln (4\sqrt{2} - 4)\right]. \qquad (62)$$

Corresponding to (48) we have

$$\sum_{1}^{\infty} \frac{(-1)^n \delta_n}{n^2} = \int_0^1 (1 - \frac{1}{\sqrt{1+x}}) \frac{\ln x}{x} dx$$

$$= 4 \int_0^{\pi/4} \sec\theta \tan\frac{\theta}{2} \ln \tan\theta \, d\theta . \quad (63)$$

8. SUMS OF THE INFINITE SERIES

$$T_m = \sum_{n=0}^{\infty} \frac{\delta_n}{(n+m+1)^3} , \quad T'_m = \sum_{n=0}^{\infty} \frac{\delta_n}{(2n+2m+1)^3} \quad (m=0,1,2,..) \quad (64)$$

Consider the integral

$$R_m = \int_0^1 \frac{t^{2m+1} \ln^2 t}{\sqrt{1-t^2}} dt . \quad (65)$$

Using (4) and applying the formula

$$\int_0^1 t^k \ln^2 t \, dt = \frac{2}{(k+1)^3} \quad (k>-1) , \quad (66)$$

we see that $R_m = \frac{1}{4} T_m$. Substituting for R_m its value given by the formula 4.261, 16. p.541 of [3] we get

$$T_m = \frac{4}{(2m+1)\delta_m} \{ [\sum_{1}^{2m+1} \frac{(-1)^k}{k} + \ln 2]^2 - $$

$$- \sum_{1}^{2n+1} \frac{(-1)^k}{k^2} - \frac{\pi^2}{12} \} . \quad (67)$$

Similarly dealing with the integral

$$R'_m = \int_0^1 \frac{t^{2m} \ln^2 t}{\sqrt{1-t^2}} dt \quad (68)$$

we find that $R'_m = 2T'_m$. Substituting for R'_m its value furnished by the formual 4.261, 15. p. 540 of [3] we obtain

$$T'_m = \frac{\pi}{4} \delta_m \{ [\sum_{1}^{2m} \frac{(-1)^k}{k} + \ln 2]^2 + \sum_{1}^{2n} \frac{(-1)^k}{k^2} + \frac{\pi^2}{12} \} . \quad (69)$$

The summation of infinite series involving powers of δ_n will be considered in a future comunication.

REFERENCES

1) BASSALI, W.A.: *"Summation of Certain Infinite Series and Definite Integrals Involving* $\delta_n = \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots (2n)}$. Rev. Téc. Ing., Univ. Zulia 10, 2(1987), 29-34.

2) CALLAN D.: *"Another way to discover that* $\sum_{1}^{\infty} \frac{\delta_n}{n} = \ln 4$". Math. Magazine 58, 5(1985), 283-284.

3) GRADSHTEYN, I.S. and RYZHIK, I.M.: *"Tables of Integrals, Series and Products"*, 4th. ed., Academic Press, New York, N.Y. (1980).

4) JOLLEY, L.B.W.: *"Summation of Series"*. Dover Publications (1961), New York.

5) KALLA, S.L. and ROSS, B.: *"The development of functional relations by means of fractional operators"*. Fractional Calculus, Pitman Pub. Ltd. (1985), 32-43.

6) KALLA, S.L. and BADER AL-SAQABI: *"A functional relation involving ψ-function"*. Rev. Téc. Ing., Univ. Zulia 8, 1 (1985), 31-35.

7) KNUTH, D.E.: *"A Catalonian Sum"*. The Amer. Math. Monthly 93, 3(1986), 220-221.

8) LEHMER, D.H.: *"Interesting series involving the central binomial coefficient"*. The Amer. Math. Monthly 92, 8(1985), 449-457.

9) ROSS, B.: *"Serendipity in Mathematics"*. The Amer. Math. Monthly 90, 8(1983), 562-566.

10) SHAFER, R.E.: The Amer. Math. Monthly 91, 10 (1984), 651.

Recibido el 14 de enero de 1988