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ON DENSITY OF FOURIER COEFFICIENTS OF A FUNCTION OF WIENER'S CLASS

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ABSTRACT

In this paper, we study the problem of density of positive and negative Fourier sine and cosine coefficients of a function of Weiner's class V_p which is a strictly larger class than the class of functions of bounded variation. In this connection we also extend a classical theorem of Fejér on the determination of the jump of a function of bounded variation to Wiener's class.

RESUMEN

En este trabajo estudiamos el problema de la densidad de los coeficientes seno y coseno de Fourier de signos positivos y negativos, de una función de la clase de Wiener V_p la cual es una clase más amplia que la clase de funciones de variación confinada. Con respecto a eso ampliamos un teorema clásico de Fejér sobre la determinación del salto de una función de variación confinada a la clase de Wiener.

1. INTRODUCTION

Let f be a real valued 2π-periodic function defined on $[0,2\pi]$ and let P: $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_n = 2\pi$ be an arbitrary partition of $[0,2\pi]$. For $1 \leq p < \infty$, we define

$$V_p(f) = \sup \{ \sum_{i=1}^{n} |f(t_i)-f(t_{i-1})|^p \}^{1/p} \quad (1)$$

where supremum has been taken over all partitions P of $[0,2\pi]$. Now we define Wiener's class by

$$V_p = \{ f: V_p(f) < \infty \} \quad (2)$$

We call $V_p(f)$ p-th variation of f. In particular V_p reduces to the class of functions of bounded variation for p=1. It is known [6] that

$V_{p_1} \subset V_{p_2}$ $(1 \leq p_1 < p_2 < \infty)$ is a strict inclusion.

Hence Wiener's class $V_p(1 < p < \infty)$ is strictly larger class than the class V_1.

2. Let $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ be the Fourier series of $f \in V_p(1 \leq p < \infty)$. We call a matrix $\Lambda = (\lambda_{n,k})(n,k = 0,1,2 \ldots)$ admissible if

$\sup_{n\geq 0} \sum_{k=1}^{\infty} |\lambda_{n,k}| = M < \infty$. It is called positive admissible if (1) $\lambda_{n,k} \geq \lambda_{n,k+1} \geq 0$ for all n and k(2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k} = 1$. Evidently every positive admissible matrix is admissible.

A sequence $\{s_k\}$ is called summable F_Λ if $\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{n,k} s_{k+s}$ exists uniformly in s. An F_Λ sumable sequence is almost convergent [4] if the matrix Λ is regular. We denote

$$q_n = \begin{cases} 1 \text{ if } b_n > 0, \\ 0 \text{ if } b_n \leq 0, \end{cases} ; \quad r_n = \begin{cases} 1 \text{ if } b_n > 0, \\ 0 \text{ if } b_n \geq 0. \end{cases}$$

And

$$\mu_n^+(s) = \sum_{k=0}^{\infty} \lambda_{n,k} q_{k+s}; \quad \mu_n^-(s) = \sum_{k=0}^{\infty} \lambda_{n,k} \gamma_{k+s} (s=0,1,2,..)$$

which are sums of positive sine coefficients and negative sine coefficients respectively. The following results on the density of Fourier sine coefficients of a function of the class V_1 are known [5].

Theorem A. Let $f \in V_1$ and let d_0 be the jump of f at zero, if f is discontinuous at zero, otherwise $d_0 = 0$. Suppose that Λ is a positive admissible matrix such that {cos kt} is summable F_Λ to zero for all $t \not\equiv 0 \pmod{2\pi}$.

(1) If $d_0 > 0$, then $\liminf_{n\to\infty} \mu_n^+(s) \geq d / V_1(f)$ uniformly in s.

(2) If $d_0 < 0$, then $\liminf_{n\to\infty} \mu_n^-(s) \geq |d_0| / V_1(f)$ uniformly in s.

(3) If $d_0 = 0$ and there is at least one value x for which the sum of jumps of f at $\pm x$ is not zero, then

$$\liminf_{n\to\infty} \mu_n^+(s) > 0 \quad \text{and} \quad \liminf_{n\to\infty} \mu_n^-(s) > 0$$

both uniformly in s.

The main aim of this paper is to extend the above Theorem A and other theorems on density into the strictly large class V_p. We first prove the following theorem which is an extension of a classical theorem of Fejér [2] on the determination of the jump of $f \in V_1$ into the class V_p. More precisely, we first prove the following theorem.

Theorem 1. Let $\Lambda = (\lambda_{nk})$ be a positive admissible matrix such that {cos kt} is summable F_Λ to zero for all $t \not\equiv 0 \pmod{2\pi}$, then for every $f \in V_p$ and for every $x \in [0,2\pi]$ the sequence

(3) $B_k(x) = \{k(b_k \cos kx - a_k \sin kx)\}$

is summable F_Λ to $\pi^{-1} d(x)$, where $d(x) = f(x+0) - f(x-0)$.

3. Now it is necessary to state a few other theorems which we shall use to prove our theorem. Young [8] proved the following two Theorems in connection with the class V_p.

Theorem B. If an $f \in V_p$ and $g \in V_q$ where $\frac{1}{p} + \frac{1}{q} > 1$, have no common points of discontinuity, their Stieltjes integral

$$\int_0^{2\pi} f \, dg$$

exists in Riemann sense.

Theorem C. If $\{f_n\} \in V_p$ $(1 \leq p < \infty)$ such that

$$V_p(f_n) \leq M$$

for all n where M is a fixed constant independent of n and $\{f_n\}$ converges to f in $[0,2\pi]$, then

$$\lim_{n\to\infty} \int_0^{2\pi} f_n \, dg = \int_0^{2\pi} f \, dg.$$

for all $g \in V_p (1 \leq p < \infty)$.

First we prove the following lemma to prove theorem 1.

Lemma 1. There exists a constant M independent of n such that

$$V_p(D_n) \leq nM$$

for all n and all $1 \leq p < \infty$ where $D_n = D_n(t) = \sum_{k=0}^{n} \cos kt$ denotes Dirchlet's kernel.

Proof of Lemma 1. It is sufficient to show that

$$V_1(D_n) \leq nM$$

for all n. Since

$$V_1(D_n) = \int_0^{2\pi} |d\, D_n(t)| dt$$

where
$$d\, D_n(t) = \frac{d}{dt}\{1 + \cos t + \cos 2t + \ldots + \cos nt\}$$
$$= -\{\sin t + 2\sin 2t + \ldots + n \sin nt\}$$

Hence

$$2 \sin \frac{t}{2}\, d\, D_n(t) = - \{\cos \frac{t}{2} + \cos \frac{3t}{2} + \ldots + n \cos \frac{(2n+1)t}{2} \}$$

and

$$4 \sin^2 \frac{t}{2}\, d\, D_n(t) = - \{(1-n) \sin nt + n \sin (n+1)t \}$$

Therefore, we can write

$$d\, D_n(t) = n \left[-G(n,t) + \frac{\sin nt}{2}\right]$$

where

$$-G(n,t) = \frac{\sin nt}{4n \sin^2 \frac{t}{2}} + \frac{\cos nt}{2 \tan \frac{t}{2}}$$

Hence

$$V_1(D_n) = n \int_0^{2\pi} \left|-G(n,t) + \frac{\sin nt}{2}\right| dt$$

$$\leq n \int_0^{2\pi} |G(n,t)|\, dt + n\pi$$

But

$$\int_0^{2\pi} |G(n,t)|\,dt = \int_0^{\pi/n} |G(n,t)|\,dt + \int_{\pi/n}^{\pi/2} |G(n,t)|\, dt + \int_{\pi/2}^{2\pi} |G(n,t)|\, dt = I_1 + I_2 + I_3.$$

It can easily be verified that

$$G(n,t) = O(n^2 t) \quad (0 \leq t \leq \frac{\pi}{n})$$

$$= O\left(\frac{1}{t}\right) \quad t > \frac{\pi}{n}.$$

Hence

$$I_1 = \int_0^{\pi/n} |G(n,t)|\,dt \leq \frac{\pi^2}{2}.$$

$$I_3 = \int_{\pi/2}^{2\pi} |G(n,t)|\,dt \leq \log 4.$$

and

$$I_2 = \int_{\pi/n}^{\pi/2} |G(n,t)|\,dt \leq \int_{\pi/n}^{\pi/2} \left| \frac{\sin nt}{4n \sin^2 \frac{t}{2}} \right| dt + \int_{\pi/n}^{\pi/2} \left|\frac{\cos nt}{2 \tan \frac{t}{2}}\right| dt = J_1 + J_2.$$

Since

$$J_1 \leq \frac{\pi^2}{4n} \int_{\pi/n}^{\pi/2} \frac{dt}{t^2} \leq \frac{\pi}{4}$$

and

$$J_2 \leq \int_{\pi/n}^{\pi/2} \frac{dt}{2} \leq \frac{\pi}{4}.$$

Collecting all the terms of I_1, I_2 and I_3, we obtain

$$V(D_n) \leq n\left(\frac{\pi^2}{2} + \frac{\pi}{2} + \log 4\right) = nM.$$

This completes the proof of Lemma 1.

4. Proof of Theorem 1. Consider the sum

$$\sum_{k=0}^{\infty} \lambda_{n,k} B_{k+s}(t) = \sum_{k=0}^{\infty} \lambda_{n,k} (k+s) \pi^{-1}$$

$$\int_0^{\pi} \psi_x(t) \sin(k+s)t \, dt$$

where $\psi_x(t) = f(x+t) - f(x-t)$. Since $\psi_x(t) \varepsilon V_p (1 \leqslant p < \infty)$ and sin kt, cos kt are continuous functions belonging to V_1, hence the integrals

$$\int_0^{\pi} \sin kt \, d\psi_x(t) \text{ and } \int_0^{\pi} \cos kt \, d\psi_x(t)$$

exist from Theorem B. Integrating by parts, we can write

$$\sum_{k=0}^{\infty} \lambda_{n,k} B_{k+s}(t) = \pi^{-1} d(x) + \pi^{-1} \int_0^{\pi} K_{n,s}(t) d\psi_x(t) \qquad (4)$$

where

$$K_{n,s}(t) = \sum_{k=0}^{\infty} \lambda_{n,k} \cos(k+s)t \qquad (5)$$

It is sufficient to show now that

$$\lim_{n \to \infty} \int_0^{\pi} K_{n,s}(t) d\psi_x(t) = 0 \qquad (6)$$

uniformly in s. Since $\psi_x(t)$ is continuous at $t = 0$, given an $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\int_0^{\delta} |d\psi_x(t)| \leqslant \frac{\varepsilon}{2M},$$

and hence

$$\left| \int_0^{\delta} K_{n,s}(t) d\psi_x(t) \right| \leqslant \frac{\varepsilon}{2}. \qquad (7)$$

Using Abel's transformation, we can write

$$K_{N,s}(t) = \sum_{k=0}^{N} \lambda_{n,k} \cos(k+s)t =$$

$$\sum_{k=0}^{N-1} \Delta\lambda_{n,k} D_{k,s}(t) + \lambda_{n,N} D_{N,s}(t)$$

where $\Delta\lambda_{n,k} = \lambda_{n,k} - \lambda_{n,k+1}$ and $D_{N,s}(t) = \sum_{k=0}^{N} \cos(k+s)t$. Hence

$$V_p(K_{N,s}(t)) \leqslant \sum_{k=0}^{N-1} |\Delta\lambda_{n,k}| V_p(D_{k,s}(t))$$

$$+ |\lambda_{n,N}| V_p(D_{N,s}(t))$$

$$\leqslant \sum_{k=0}^{N} |\Delta\lambda_{n,k}| V_p(D_{k,s}(t))$$

But from lemma 1, we conclude that there exists a constant M independent of N and s such that

$$V_p(K_{N,s}(t)) \leqslant M \sum_{k=0}^{N} k |\Delta \lambda_{n,k}|$$

for $p > 1$. Using the definition of a positive admissible matrix and taking limit as $N \to \infty$, we obtain

$$V_p(K_{n,s}(t) \leqslant M$$

Now using Theorem C for a given $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$\left|\int_{\delta}^{\pi} K_{n,s}(t)d\psi_x(t)\right| \leqslant \frac{\varepsilon}{2}. \qquad (8)$$

From (7) and (8), we obtain (6) which is sufficient to prove Theorem 1.

Remark 1. If $\Lambda = (\lambda_{n,k})$ is a positive admissible matrix, then similarly we can further prove that under the hypothesis of Theorem 1, not only the sequence $\{B_k(x)\}$ but even $\{|B_k(x)|\}$ is summable F_Λ to $\pi^{-1}d(x)$ for every $x \in [0,2\pi]$ and for every $f \in V_p(1 \leqslant p < \infty)$.

Theorem 1 contains as a special case the following sharpened version of Fejér's Theorem (cf. Zygmund [9] p. 107, Th. 9.3.).

Corollary 1. If $f \in V_1$, then

$$\lim_{n\to\infty}(n+1)^{-1}\sum_{k=-n-s}^{n+s} k(b_k\cos kx + a_k\sin kx) = \pi^{-1}d(x)$$

uniformly in s for every $x \in [0,2\pi]$

5. Applying Theorem 1, we extend Theorem A into Wiener's class V_p in the following form.

Theorem 2. Let $f \in V_p(1 < p < \infty)$ and let d_0 be the jump of f, if f is discontinuous at zero, otherwise $d_0 = 0$. Suppose that $\Lambda = (\lambda_{n,k})$ is a positive admissible matrix such that $\{\cos kt\}$ is summable F_Λ to zero for all $t \not\equiv 0 \pmod{2\pi}$.

(1) If $d_0 > 0$, then $\lim\limits_{n\to\infty}\inf \mu_n^+(s) \geqslant d_0/2^{1/q}\, V_p(f)$ uniformly in s.

(2) If $d_0 < 0$, then $\lim\limits_{n\to\infty}\inf \mu_n^-(s) \geqslant |d_0|2^{1/q}\, V_p(f)$ uniformly in s.

(3) If $d_0 = 0$, then $\underline{\lim}_{n\to\infty}\inf \mu_n^+(s) \geqslant 0$ and $\underline{\lim}_{n\to\infty}\inf \mu_n^-(s) \geqslant 0$ uniformly in s, where $\frac{1}{p} + \frac{1}{q} = 1$.

We need the following theorem [cf. Siddiqi [6] p. 569] in which we calculate the order of Fourier coefficients of a function

$$f \in V_p\ (1 \leqslant p < \infty).$$

Fourier coefficients of a function $f \in V_p(1 \leqslant p < \infty)$

Theorem D. If $f \in V_p(1 \leqslant p < \infty)$, then

$$|a_n|,\ |b_n| \leqslant \frac{2^{1/q}\, V_p(f)}{\pi\, n^{1/p}}$$

for all $n \geqslant 1$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Using Theorem 1 and Theorem D, we give the proof of Theorem 2 below.

Proof of Theorem 2. Since $(k+s)^{1/p} \leqslant (k+s)$ for every p and

$$\sum_{k=0}^{\infty} \lambda_{n,k}(k+s)(b_{k+s}\cos(k+s)x - a_{k+s}\sin(k+s)x) =$$

$\pi^{-1}d(x) + o(1)\ (n \to \infty)$ for every $f \in V_p$ and for every $x \in [0,2\pi]$ from Theorem 1, hence we obtain

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} \lambda_{n,k}(k+s)^{1/p}(b_{k+s}\cos(k+s)x - a_{k+s}\sin(k+s)x) =$$

$$= \pi^{-1}d(x)$$

uniformly in s for very $x \in [0,2\pi]$. Hence for $x = 0$, we obtain

$$\sum_{k=0}^{\infty} \lambda_{nk}(k+s)^{1/p}b_{k+s} = \pi^{-1}d_0 + o(1) \quad (n \to \infty) \qquad (9)$$

uniformly in s. We can write

$$\sum_{k=0}^{\infty} \lambda_{n,k}(k+s)^{1/p} b_{k+s} = \sum{}^{+} + \sum{}^{-}$$

where Σ^{+} and Σ^{-} denote positive sum and negative sum respectively. Since b_{k+s} are Fourier sine coefficients of $f \in V_p$, hence from Theorem D

$$-\frac{2^{1/q}}{\pi} V_p(f) \leq (k+s)^{1/p} b_{k+s} \leq \frac{2^{1/q}}{\pi} V_p(f) \qquad (10)$$

where $V_p(f)$ is the p-th variation of f. Now using the definition of $\mu_n^{+}(s)$ and (10), we obtain

$$\sum_{k=0}^{\infty} \lambda_{n,k}(k+s)^{1/p} b_{k+s} \leq \sum{}^{+} \lambda_{n,k}(k+s)^{1/p} b_{k+s}$$

$$\leq \mu_n^{+}(s) \frac{2^{1/q}}{\pi} V_p(f)$$

Taking limit as $n \to \infty$ and using (9), we obtain

$$\pi^{-1} d_0 \leq \lim_{n\to\infty} \inf \mu_n^{+}(s) \frac{2^{1/q}}{\pi} V_p(f)$$

which can be interpreted as

$$\lim_{n\to\infty} \inf \mu_n^{+}(s) \geq d_0/2^{1/q} V_p(f)$$

uniformly in s which is case I of Theorem 2.

Case 2. If we apply the same arguments of Case 1 on $-f$ instead of f, we obtain

$$\lim_{n\to\infty} \inf \mu_n^{-}(s) \geq |d_0|/2^{1/q} V_p(f)$$

uniformly in s.

Case 3. From Remark 1, $\{|B_k(x)|\}$ is summable F_Λ to $\pi^{-1} d(x)$ for all $x \in [0,2\pi]$. Hence for $x = 0$, we obtain

$$\sum_{k=0}^{\infty} \lambda_{n,k}(k+s)^{1/p} |b_{k+s}| = \pi^{-1} d_0 + o(1) \quad (n \to \infty) \qquad (11)$$

uniformly in s. Adding (9) and (11) and using the definition of absolute value, we obtain

$$2\sum{}^{+} \lambda_{n,k}(k+s)^{1/p} b_{k+s} = 2\pi^{-1} d_0 + o(1) \quad (n \to \infty). \qquad (12)$$

Now from (10) and (12) and by the definition of $\mu_n^{+}(s)$ we obtain

$$\lim_{n\to\infty} \inf \mu_n^{+}(s) \geq 0$$

uniformly in s which is first relation of case 3. Similarly we can prove the second relation. Hence Theorem 2 is completely proved.

We note that if if we choose Λ as a matrix of arithmetic mean and $p = 1$, $s = 0$, our Theorem 2 gives a sharpened version of a Theorem of M and S. Izumi [3] (cf. Siddiqi [5] p. 94, Th. A'). Now we define

$$q_n(x) = \begin{cases} 1 & \left(\frac{x}{\pi} < n^{1/p} b_n \leq 2^{1/q} \frac{V_p(f)}{\pi}\right); \\ 0 & \text{otherwise} \end{cases}$$

$$r_n(x) = \begin{cases} 1 & \left(-2^{1/q} \frac{V_p(f)}{\pi} \leq n^{1/p} b_n < \frac{x}{\pi}\right); \\ 0 & \text{otherwise} \end{cases}$$

We also define

$$\mu_n^{+}(s)(x) = \sum_{k=0}^{\infty} \lambda_{n,k} q_{k+s}(x)$$

$$\mu_n^{-}(s)(x) = \sum_{k=0}^{\infty} \lambda_{n,k} r_{k+s}(x)$$

Then we can similarly (cf. [5], p. 100) prove the following :

Theorem 3. Let $f \in V_p(1 \leq p < \infty)$ and let d_0 be the jump of f at zero, if f is discontinuous at zero, otherwise $d_0 = 0$. Suppose that Λ is a positive admissible matrix such that {cos kt} is summable F_Λ to zero for all $t \not\equiv 0 \pmod{2}$.

(1) If $d_0 > x$ then

$$\liminf_{n\to\infty} \mu_n^+(s)(x) \geq \frac{|(d_0-x)|}{(V_p(f)-|d_0|+|d_0-x|+|x|)2^{1/q}}$$

unformly in s.

2) If $d_0 < x$ then

$$\liminf_{n\to\infty} \mu_n^-(s)(x) \geq \frac{|(d_0-x)|}{(V_p(f)-|d_0|+|d_0-x|+|x|)2^{1/q}}$$

As a special case for x = 0, Theorem 3 reduces to Theorem 2.

6. Now we consider the problem of density of Fourier cosine coefficients a_n of a function of the class V_p. We denote

$$q_n^* = \begin{cases} 1 & \text{if } a_n > 0 \\ 0 & \text{if } a_n \leq 0 \end{cases} ; \quad r_n^* = \begin{cases} 1 & \text{if } a_n < 0, \\ 0 & \text{if } a_n \geq 0; \end{cases}$$

and also denote

$$\nu_n^+(s) = \sum_{k=0}^{\infty} \lambda_{n,k} q^*_{k+s}, \quad \nu_n^-(s) =$$

$$\sum_{k=0}^{\infty} \lambda_{n,k} r^*_{k+s} \quad (s = 0,1,\ldots)$$

then we prove the following :

Theorem 4. Let $f \in V_p(1 < p < \infty)$ and has points of discontinuity different than origin. Suppose that $\Lambda = (\lambda_{n,k})$ is a positive admissible matrix such that {sin kt} is summable F_Λ to zero for all $t \not\equiv 0 \pmod{2\pi}$ then

$$\liminf_{n\to\infty} \nu_n^+(s) \geq 0$$

uniformly in s and also

$$\liminf \nu_n^-(s) \geq 0$$

uniformly in s.

For the proof of the above theorem, we need the following lemma.

Lemma 2. Let $\Lambda = (\lambda_{n,k})$ be an admissible matrix such that {sin kt} is summable F_Λ to zero for all $t \not\equiv 0 \pmod{2\pi}$, then for every $f \in V_p$ $(1 < p < \infty)$ and for every $x \in [0,2]$, the sequence

$$\{A_k(x)\} = \{k(a_k \cos kx + b_k \sin kx)\}$$

is summable F_Λ to zero.

The proof of Lemma 2 is similar to the proof of Theorem 1. Hence we shall not give the proof of lemma 2 here.

Proof of Theorem 4. Under the hypothesis of Theorem 4, $\{A_k(x)\}$ is summable F_Λ to zero for every $x \in [0,2\pi]$. Hence for x = 0, we obtain

$$\sum_{k=0}^{\infty} \lambda_{n,k}(k+s)a_{k+s} = o(1) \quad (n \to \infty)$$

uniformly in s. Since Λ is a positive matrix, hence

$$\sum_{k=0}^{\infty} \lambda_{n,k}(k+s)^{1/p} a_{k+s} = o(1) \quad (n \to \infty) \quad (13)$$

uniformly in s. Since a_{k+s} are Fourier cosine coefficients of a function $f \in V_p$, hence from Theorem E we obtain

$$\frac{-2^{1/q}}{\pi} V_p(f) \leq (k+s)a_{k+s} \leq \frac{2^{1/q}}{\pi} V_p(f). \qquad (14)$$

Now using the definition of $\overset{+}{\nu}_n(s)$ and (14) we obtain

$$\sum_{n\to\infty}^{\infty} \lambda_{n,k}(k+s)^{1/p} a_{k+s} \leq \sum \lambda_{n,k}(k+s)^{1/p} a_{k+s}$$

$$\leq \frac{2^{1/q}}{\pi} V_p(f)\overset{+}{\nu}_n(s).$$

Taking limit as $n\to\infty$ and using (13), we obtain

$$\liminf_{n\to\infty} \overset{+}{\nu}_n(s) \geq 0$$

uniformly in s. Similarly we can prove the second relation of this Theorem. Hence Theorem 4 is completely proved.

If we choose Λ to be the matrix of arithmetic mean, $p = 1$ and $s = 0$, then our Theorem 4 gives a sharpened version of another Theorem of M and S Izumi [3].

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