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R.K. Saxena y G.C. Modi
Department of Mathematics and Statistics
University of Jodhpur
Jodhpur-342001, Rajasthan, India

S.L. Kalla
División de Post grado, Facultad de Ingeniería
Universidad del Zulia
Maracaibo-Venezuela

A BASIC ANALOGUE OF FOX'S H-FUNCTION

*(Al Libertador Simón Bolívar,
en el bicentenario de su nacimiento)*

ABSTRATC

Certain basic integrals and integral representations for a basic analogue of Fox's H-function are investigated in this paper.

RESUMEN

En este trabajo estudiamos un análogo básico de la función H- de Fox. Obtenemos algunas integrales y representaciones integrales de la función.

INTRODUCTION

The object of this paper is to define a basic analogue of Fox's H-function and to establish some of its fundamental properties. The results proved are of general character and include, as **special** cases, the results given earlier by Agarwal.

DEFINITION OF A BASIC FOX'S H_q-FUNCTION

We define a basic analogue of Fox's H-function as

$$H^{m_1,n_1}_{A,B}\left[z;q\,\middle|\,\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(b_j-\beta_j s)\prod_{j=1}^{n_1} G(1-a_j+\alpha_j s)\pi z^s\,ds}{\prod_{j=m_1+1}^{B} G(1-b_j+\alpha_j s)\prod_{j=n_1+1}^{A} G(a_j-\alpha_j s)G(1-s)\sin\pi s}$$

where $0 \leqslant m_1 \leqslant B$, $0 \leqslant n_1 \leqslant A$; α_i's and $_j$'s are all positive, the contour C is a line parallel to Rl (ωs), with indentations if necessary, in such a manner that all poles of $G(b_j-\beta_j s)$, $1 \leqslant j \leqslant m_1$ are to the right, and those of $G(1-a_j+\alpha_j s)$, $1 \leqslant j \leqslant m_1$, to the left of C. The integral converges if $Rl\{s\log(z) - \log\sin\pi s\}<0$ for large values of $|s|$ on the contour i.e., if $|\{\arg(z)-\omega_2\omega_1^{-1}\log|z|\}|<\pi$.

If we set $\alpha_j = \beta_i = 1$, $1\leqslant j\leqslant A$, $1\leqslant i\leqslant B$ in (2.1), then it reduces to the basic analogue of Meijer's G-function, namely

$$G^{m_1,n_1}_{A,B}\left[z;q\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right] = H^{m_1,n_1}_{A,B}\left[z;q\,\middle|\,\begin{matrix}(a,1)\\(b,1)\end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(b_j-s)\prod_{j=1}^{n_1} G(1-a_j+s)\pi z^s\,ds}{\prod_{j=m_1+1}^{B} G(1-b_j+s)\prod_{j=n_1+1}^{A} G(a_j-s)G(1-s)\sin\pi s} \quad (2.2)$$

If we put $m_1 = B$, $n_1 = 0$; $\beta_j = \alpha_j = 1$, $1< i <A$, $1\leqslant j\leqslant B$; in (2.1) then it reduces to an E_q-function due to Agarwal [1], which itself is a generalization of basic analogue of MacRobert's E_q-function given earlier by Agarwal [2].

From the definition (2.1), it readily follows that

$$z^\sigma H^{m_1,n_1}_{A,B}\left[z;q\,\middle|\,\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] =$$

$$(-1)^\sigma H^{m_1+1,n_1}_{A+1,B+1}\left[z;q\,\middle|\,\begin{matrix}(a_j+\alpha_j\sigma,\alpha_j),(1+\sigma,1)\\(1,1),\ (b_j+\beta_j\sigma,\beta_j)\end{matrix}\right] \quad (2.3)$$

and

$$H^{m_1,n_1}_{A,B}\left[z^{-1};q\,\middle|\,\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] =$$

$$-H^{n_1+1,m_1}_{B,A+2}\left[z;q\,\middle|\,\begin{matrix}(1-b_j,\beta_j)\\(1,1),(1-a_j,\alpha_j),(0,1)\end{matrix}\right] \quad (2.4)$$

CERTAIN BASIC INTEGRALS INVOLVING H_q-FUNCTION

The following basic integrals are to be established here.

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1} E_q(qx) H^{m_1,n_1}_{A,\,B}\left[zx^{-\rho};q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] d(q,x) = H^{m_1+1,n_1}_{A,\,B+1}\left[z;q \,\middle|\, \begin{matrix}(a,\alpha)\\(\sigma,\rho),(b,\beta)\end{matrix}\right], \qquad (3.1)$$

for $Rl(\sigma) > 0$ and $Rl(\rho) > 0$, $|\{\arg z - \omega_2\omega_1^{-1}\log|z|\}| < \pi$

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1}(1-qx)_{\rho-\sigma-1}\, H^{m_1,n_1}_{A,\,B}\left[zx^{-u};q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] d(q,x)$$

$$= G(\rho-\sigma) H^{m_1+1,n_1}_{A+1,B+1}\left[z;q \,\middle|\, \begin{matrix}(a,\alpha),\ (\sigma,u)\\(\rho,u),\ (b,\beta)\end{matrix}\right], \qquad (3.2)$$

for $Rl(\sigma) > 0$, $Rl(\rho) > 0$, $Rl(u) > 0$, $Rl(\rho-\sigma) > 0$, $|\{\arg z - \omega_2\omega_1^{-1}\log|z|\}| < \pi$.

$$\frac{1}{2\pi i}\int_C e_q(x)x^{-\sigma} H^{m_1,n_1}_{A,\,B}\left[zx^{\rho};q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] dx$$

$$= G(1) H^{m_1,n_1}_{A+1,B}\left[z;q \,\middle|\, \begin{matrix}(a,\alpha),\ (\sigma,\rho)\\(b,\beta)\end{matrix}\right], \qquad (3.3)$$

where the path of integration C encircles the null-point and also in the usual manner, can be deformed into a loop parallel to the imaginary axis.

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1} E_q(qx) H^{m_1,n_1}_{A,\,B}\left[zx^{\rho};q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] d(q,x)$$

$$= H^{m_1,n_1+1}_{A+1,\,B}\left[z;q \,\middle|\, \begin{matrix}(1-\sigma,\rho),(a,\alpha)\\(b,\beta)\end{matrix}\right], \qquad (3.4)$$

for $Rl(\sigma) > 0$, $Rl(\rho) > 0$ and $|\arg z - \omega_2\omega_1^{-1}\log|z||< \pi$.

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1}(1-qx)_{\rho-\sigma-1}\, H^{m_1,n_1}_{A,\,B}\left[zx^{u};q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] d(q,x) =$$

$$G(\rho-\sigma) H^{m_1,n_1+1}_{A+1,B+1}\left[z;q \,\middle|\, \begin{matrix}(1-\rho,u,(1,\alpha)\\(b,\beta),(1-\sigma,u)\end{matrix}\right], \qquad (3.5)$$

for $Rl(\sigma) > 0$, $Rl(u) > 0$, $Rl(\rho-\sigma) > 0$, $|\{\arg z - \omega_2\omega_1^{-1}\log|z|\}| < \pi$

$$\frac{1}{2\pi i}\int_C e_q(x)x^{-\sigma} H^{m_1,n_1}_{A,\,B}\left[zx^{-\rho};q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right] dx$$

$$= G(1) H^{m_1,n_1}_{A,B+1}\left[z;q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta),\ (1-\sigma,\rho)\end{matrix}\right] \qquad (3.6)$$

Proof: In view of the definition (2.1), the expression on the left of (3.1), can be written as

$$\frac{G(1)}{(1-q)}\int_0^1 x^{\sigma-1} E_q(qx)\ \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(b_j-\beta_j s)\prod_{j=1}^{n_1} G(1-a_j+\alpha_j s)}{\prod_{j=m_1+1}^{B} G(1-b_j+\beta_j s)\prod_{j=n_1+1}^{A} G(a_j-\alpha_j s)}$$

$$X\ \frac{\pi\, z^s\, x^{-\rho s}\, ds}{G(1-s)\sin \pi s}\ d(q,x)$$

If we interchange the order of integration, which is valid for $Rl(\sigma) > 0$, and $|\{\arg z - \omega_2\omega_1^{-1}\log|z|\}| < \pi$, then the above expression reduces to

$$\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(b_j-\beta_j s)\prod_{j=1}^{n_1} G(1-a_j+\alpha_j s)\,\pi 3^s}{\prod_{j=m_1+1}^{B} G(1-b_j+\beta_j s)\prod_{j=n_1+1}^{A} G(a_j-\alpha_j s)G(1-s)\sin \pi s}$$

$$X\left\{\frac{G(1)}{1-q}\int_0^1 x^{\rho-\sigma s-1}\ E_q(qx)d(q,x)\right\} ds.$$

On evaluating the inner integral with help of an integral due to Hahn [3] and interpreting the result thus obtained with the help of (2.1), we arrive at the desired result (3.1)

The remaining integrals (3.2) to (3.6) can be evaluated in the same way by making use of the results (3.16) and ξ 9(b) given earlier by Hahn ([3]).

CERTAIN INTEGRALS INVOLVING

BASIC MEIJER'S G-FUNCTION

If we specialize the parameters in (3.1) to (3,6) and make use of the result (2.2), the following results are obtained.

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1} E_q(qx)\, G^{m_1,n_1}_{A,\,B}\left[zx^{-u};q\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right] d(q,x)$$

$$= \frac{G(\sigma)}{\prod_{i=1}^{u} G\left(q'^{\frac{\sigma+i-1}{u}}\right)}\, G'^{\,m_1+u,n_1}_{A,B+u}\left[z;q\,\middle|\,\begin{matrix}(a)\\(c),(b)\end{matrix}\right], \qquad (4.1)$$

where $q' = q^u$, u is a positive integer, $Rl(\sigma) > 0$, $c_i = q'^{\frac{\sigma+i-1}{u}}$ for $i = 1,\dots,u$, $|\{\arg z - \omega_2\omega_1^{-1}\log|z|\}| < \pi$.

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1}(1-qx)_{\rho-\sigma-1}\, G^{m_1,n_1}_{A,\,B}\left[zx^{-u};q\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right] d(q,x)$$

$$= \frac{G(\rho-\sigma)G(\sigma)}{G(\rho)} \prod_{i=1}^{u} \frac{G\left(q'^{\frac{\rho+i-1}{u}}\right)}{G\left(q'^{\frac{\sigma+i-1}{u}}\right)}$$

$$G'^{\,m_1+u,n_1}_{A+u,B+u}\left[z;q\,\middle|\,\begin{matrix}(a),(d)\\(c),(b)\end{matrix}\right], \qquad (4.2)$$

where $q' = q^u$, u is a positive integer, $Rl(\rho) > Rl(\sigma) > 0$, $d_i = q'^{\frac{\sigma+i-1}{u}}$, $c_i = q'^{\frac{\rho+i-1}{u}}$ for $i = 1,\dots,u$, $|\{\arg z - \omega_2\omega_1^{-1}\log|z|\}| < \pi$.

$$\frac{1}{2\pi i}\int_C e_q(x)x^{-\sigma}\, G^{m_1,n_1}_{A,\,B}\left[zx^{u};q\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right] dx$$

$$= \frac{G(1)\prod_{i=1}^{u} G\left(q'^{\frac{\delta+i-1}{u}}\right)}{G(\delta)}\, G'^{\,m_1,n_1}_{A+u,B}\left[z;q\,\middle|\,\begin{matrix}(a),(c)\\(b)\end{matrix}\right], \qquad (4.3)$$

where the path of integration is the same as in the case of the integral (3.3), $Rl(\sigma) > 0$, $c_i = q'^{\frac{\sigma+i-1}{u}}$, $i = 1,\dots,u$; $q' = q^u$.

$$\frac{G(1)}{(1-q)}\int_0^1 x^{\sigma-1} E_q(qx)\, G^{m_1,n_1}_{A,\,B}\left[zx^{u};q\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right] d(q,x)$$

$$= \frac{\prod_{i=1}^{u} G\left(q'^{\frac{i-\sigma}{u}}\right)}{G(1-\sigma)}\, G'^{\,m_1,n_1+u}_{A+u,B}\left[z;q\,\middle|\,\begin{matrix}(c),(a)\\(b)\end{matrix}\right], \qquad (4.4)$$

where $q' = q^u$, u is a positive integer, $Rl(\sigma) > 0$, $c_i = q^{\frac{i-\sigma}{u}}$ for $i = 1,\dots,u$.

$$\frac{G(1)}{1-q}\int_0^1 x^{\sigma-1}(1-qx)_{\rho-\sigma-1}\, G^{m_1,n_1}_{A,\,B}\left[zx^{u};q\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right] d(q,x)$$

$$= \frac{G(\rho-\sigma)G(1-\sigma)}{G(1-\rho)} \prod_{i=1}^{u} \frac{G\left(q'^{\frac{-\rho+i}{u}}\right)}{G\left(q'^{\frac{-\sigma+i}{u}}\right)}$$

$$G'^{\,m_1+u,n_1}_{A+u,B+u}\left[z;q\,\middle|\,\begin{matrix}(a),(d)\\(c),(b)\end{matrix}\right],$$

where $q' = q^u$, u is a positive integer, $Rl(\rho) > Rl(\sigma) > 0$, $d_i = q^{\frac{i-\sigma}{u}}$, $c_i = q^{\frac{i-\rho}{u}}$, for $i = 1,\dots,u$.

$$\frac{1}{2\pi i}\int_C e_q(x)X^{-\sigma}\, G^{m_1,n_1}_{A,\,B}\left[zx^{-\rho};q\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right] dx$$

$$= \frac{G(1)G(1-\sigma)}{\prod_{i=1}^{\rho} G\left(q'^{\frac{i-\sigma}{\rho}}\right)}\, G'^{\,m_1,n_1}_{A,B+\rho}\left[z;q\,\middle|\,\begin{matrix}(a)\\(b),(c)\end{matrix}\right], \qquad (4.6)$$

where $q' = q^{\rho}$, ρ is a positive integer, $Rl(\sigma) > 0$, $c_i = q^{\frac{i-\sigma}{\rho}}$, for $i = 1,\dots,\rho$.

It is interesting to observe that if we take $m_1 = B$, $n_1 = 0$ in (4.1), (4.2) and (4.3); and use the relation

$$G^{B,0}_{A,B}\left[z;q\,\middle|\,\begin{matrix}(a)\\(b)\end{matrix}\right] = E_q\left[B;b_p:A;a_t:z\right]$$

then (4.1), (4.2) and (4.3) respectively give rise to the results (6.1) - (6.3) due to Agarwal [1].

INTEGRAL REPRESENTATION FOR BASIC FOX'S H_q-FUNCTION

$$H^{m_1,n_1}_{A,\,B}\left[z,q\,\middle|\,\begin{matrix}(a,\alpha)^*\\(b_1,1),(b_2,1),(b,\beta)^*\end{matrix}\right]$$

$$= G(b_1)\prod_{j=m_1+1}^{B}\frac{1}{G(1)2\pi i}\int_{C_j} e_q(x_j)$$

$$X\, x_j^{-b_j}\, dx_j\Big\}\prod_{j=n_1+1}^{A}\left\{\frac{1}{G(1)2\pi i}\int_{C_j} e_q(\lambda_j)\lambda_j^{-a_j}\, d\lambda_j\right\}$$

$$\prod_{j=1}^{n_1}\left\{\frac{G(1)}{(1-q)}\int_0^1 \lambda_j^{a_j-1}\ X\ E_q(q\lambda_j)d(q,\lambda_j)\right.\ \prod_{j=3}^{m_1}$$

$$\cdot\left\{\frac{G(1)}{1-q}\int_0^1 E_q(qu_j)u_j^{b_j-1}\,d(q,u_j)\right\}\frac{1}{1-q}\int_0^1 u_2^{b_2-1}\times E_q(qu_2)\,{}_1\phi_0\left(\begin{matrix}b_1;\\ -;\end{matrix}\; -u_2\prod_{j=1}^{A}\lambda_j^{-\alpha_j}\prod_{j=3}^{B}u_j^{\beta_j}/z\right)d(q,u_2), \quad (5.1)$$

where $(a,\alpha)^*$ and $(b,\beta)^*$ denote the sequence of A and B-2 pairs $(1-a_1,\alpha_1),\ldots,(1-a_{n_1},\alpha_{n_1}),(a_{n_1+1},\alpha_{n_1+1}),\ldots,(a_A,\alpha_A);(b_3,\beta_3),\ldots,(b_{m_1},\beta_{m_1}),(1-b_{m_1+1},\beta_{m_1+1}),\ldots,(1-b_B,\beta_B)$; $Rl(b_j)>0$, $2\leqslant j\leqslant B$; $Rl(a_j)>0$, $1\leqslant j\leqslant A$.

$$H^{m,n}_{m+n-2,m+n}\left[\begin{matrix}(1-a_1,\alpha_1),\ldots,(1-a_n,\alpha_n),(c_1,\beta_1),\ldots,(c_{m-2},\beta_{m-2})\\ (b_1,\beta_1),\ldots,(b_m,\beta_m),(1-d_1,\alpha_1),\ldots,(1-d_n,\alpha_n)\end{matrix}\right] = G(b_m)\prod_{j=1}^{n}\left\{\frac{G(1)}{1-q}\int_0^1\lambda_j^{a_j-1}(1-q\lambda_j)_{d_j-a_j-1}\,d(q,\lambda_j)/G(d_j-a_j)\right\}\prod_{k=1}^{m-2}\left\{\frac{G(1)}{1-q}\int_0^1 u_k^{b_k-1}(1-qu_k)_{c_k-b_k-1}\,d(q,u_k)/G(c_k-b_k)\right\}\frac{1}{1-q}\int_0^1 u_{m-1}^{b_{m-1}-1}E_q(qu_{m-1})\,{}_1\phi_0\left(\begin{matrix}b_m;\\ -;\end{matrix}\; -\frac{z}{u_{m-1}}\prod_{j=1}^{n}\lambda_j^{\alpha_j}\prod_{j=1}^{m-2}u_j^{-\beta_j}\right)d(q,u_{m-1}), \quad (5.2)$$

where $Rl(d_j) > Rl(a_j) > 0$, $Rl(c_k) > Rl(b_k) > 0$, $Rl(\alpha_j) > 0$, $\beta_m=\beta_{m-1}=1$, $Rl(\beta_k) > 0$, $1\leqslant j\leqslant n$; $1\leqslant k\leqslant m-2$; α's and β's are all positive quantities.

$$H^{m_1,A-m_1}_{A-2,\,B}\left[z;q\left|\begin{matrix}(1-a_{m+1},\alpha_{m+1}),\ldots,(1-a_A,\alpha_A),\ (a_3,\beta_3),\ldots,(a_m,\beta_m)\\ (b_1,1),(b_2,1),(b_3,\alpha_3),\ldots,\ (b_m,\alpha_m),(1-b_{m_1+1},\beta_{m+1}),\ldots,(1-b_B,\beta_B)\end{matrix}\right.\right] = G(b_1)\prod_{j=m_1+1}^{B}\left\{\frac{1}{G(1)2\pi i}\int_{C_j}e_q(u_j)u_j^{-b_j}du_j\right\}\prod_{k=3}^{m_1}\left\{\frac{G(1)}{1-q}\int_0^1\lambda_k^{b_k-1}(1-q\lambda_k)_{a_k-b_k-1}\,d(q,\lambda_k)/G(a_k-b_k)\right\}\prod_{t=m+1}^{A}\left\{\frac{G(1)}{1-q}\int_0^1 x_t^{a_t-1}E_q(qx_t)\,d(q,x_t)\right\}\frac{1}{1-q}\int_0^1\lambda_2^{b_2-1}E_q(q\lambda_2)\,{}_1\phi_0\left(\begin{matrix}b_1;\\ -;\end{matrix}\; -z\prod_{j=m_1+1}^{B}u_j^{-\beta_j}\prod_{K=3}^{m_1}\lambda_k^{\beta_k}\prod_{t=m_1+1}^{A}x_t^{\alpha_t}/\lambda_2\right)d(q,\lambda_2) \quad (5.3)$$

where $Rl(a_j) > 0$, $Rl(a_k) > Rl(b_k) > 0$, $3\leqslant k\leqslant m_1$, $m_1+1\leqslant j\leqslant A$, $Rl(b_t) > 0$, $m_1+1\leqslant t\leqslant B$; α's and β's are all positive quantities.

Proof: To prove (5.1) we start with the known result (2 ; (2.1))

$$E_q(a,b::z) = \frac{G(a)}{1-q}\int_0^1 E_q(q\lambda)\lambda^{b-1}\,{}_1\phi_0(a;-\lambda/z)\,d(q,\lambda)$$

$$= \frac{1}{2\pi i}\int_C \frac{G(a-s)G(b-s)\pi z^s ds}{G(1-s)\sin\pi s},$$

where the contour C is a line parallel to $Rl(\omega s)=0$. The integral converges if $Rl[s\log(z) - \log\sin\pi s] < 0$ for large values of $|s|$ on the contour, i.e., if $|\{\arg(z)-\omega_2\omega_1^{-1}\log|z|\}| < \pi$.

The expression of the right of (5.1) can be written as

$$\prod_{j=m_1+1}^{B}\left\{\frac{1}{G(1)2\pi i}\int_{C_j}e_q(u_j)u_j^{-b_j}du_j\right\}\prod_{j=n_1+1}^{A}\frac{1}{G(1)2\pi i}\int_{C_j}e_q(\lambda_j)\lambda_j^{-a_j}d\lambda_j\Bigg\}\prod_{j=1}^{n_1}\left\{\frac{G(1)}{1-q}\int_0^1\lambda_j^{a_j-1}E_q(q\lambda_j)d(q,\lambda_j)\right\}$$

$$\prod_{j=3}^{m_1}\left\{\frac{G(1)}{1-q}\int_0^1 E_q(qu_j)u_j^{b_j-1}d(q,u_j)\right\}\frac{1}{2\pi i}\int_C\frac{G(b_1-s)G(b_2-s)\pi z^s}{G(1-s)\sin\pi s}\prod_{j=3}^{B}u_j^{-\beta_j s}\prod_{j=1}^{A}\lambda_j^{\alpha_j s}\,ds = \prod_{j=m_1+1}^{B}\left\{\frac{1}{G(1)2\pi i}\int_{C_j}e_q(u_j)u_j^{-b_j}du_j\right\}\prod_{j=n_1+1}^{A}\frac{1}{G(1)2\pi i}\left\{\int_{C_j}e_q(\lambda_j)\lambda_j^{-a_j}d\lambda_j\right\}$$

$$\prod_{j=1}^{n_1}\left\{\frac{G(1)}{1-q}\int_0^1\lambda_j^{a_j-1}E_q(q\lambda_j)d(q,\lambda_j)\right\}\prod_{j=4}^{m_1}\frac{G(1)}{1-q}\int_0^1 E_q(qu_j)u_j^{b_j-1}d(q,u_j)\Bigg\}\frac{G(1)}{1-q_A}\int_0^1 E_q(qu_3)u_3^{b_3-1}d(q,u_3)$$

$$\frac{1}{2\pi i}\int_C\frac{G(b_1-s)G(b_2-s)\pi z^s\prod_{j=1}\lambda_j^{\alpha_j s}\,ds}{G(1-s)\sin\pi s\prod_{j=3}^{B}u_j^{\beta_j s}}$$

On changing the order of integration which is valid by absolute convergence of both the integrals, for $Rl(b_3) > 0$ and $|\{\arg z - \omega_2\omega_1^{-1}\log|z|\}| < \pi$; the R.H.S. for the above expression becomes

$$\prod_{j=m_1+1}^{B}\left\{\frac{1}{G(1)2\pi i}\int_{C_j}e_q(u_j)u_j^{-b_j}du_j\right\}\prod_{j=n_1+1}^{A}\left\{\frac{1}{G(1)2\pi i}\right.$$

$$\int_{C_j} e_q(\lambda_j)\lambda_j^{-a_j}\, d\lambda_j\Big\} \prod_{j=1}^{n_1}\Big\{\frac{G(1)}{1-q}\int_0^1 \lambda_j^{a_j-1} E_q(q\lambda_j)d(q,\lambda_j)\Big\}\prod_{j=4}^{m_1}\Big\{\frac{G(1)}{1-q}\int_0^1 E_q(qu_j)u_j^{b_j-1}\, d(q,u_j)\Big\}\frac{1}{2\pi i}\int_C \frac{G(b_1-s)G(b_2-s)\pi z^s \prod_{j=1}^{A}\lambda_j^{\alpha_j s}}{G(1-s)\sin\pi s \prod_{j=4}^{B} u_j^{\beta_j s}}\Big\{\frac{G(1)}{1-q}\int_0^1 E_q(qu_3)u_3^{b_3-\beta_3 s-1}\, d(q,u_3)\Big\}\, ds$$

On evaluating the innermost integral, it gives

$$\prod_{j=m_1+1}^{B}\Big\{\frac{1}{G(1)2\pi i}\int_{C_j} e_q(u_j)u_j^{-b_j}\,du_j\Big\}\prod_{j=n_1+1}^{A}\Big\{\frac{1}{G(1)2\pi i}\int_{C_j} e_q(\lambda_j)\lambda_j^{-a_j}\,d\lambda_j\Big\}\prod_{j=1}^{n_1}\Big\{\frac{G(1)}{1-q}\int_0^1\lambda_j^{a_j-1}E_q(q\lambda_j)d(q,\lambda_j)\Big\}$$

$$\prod_{j=4}^{m_1}\Big\{\frac{G(1)}{1-q}\int_0^1 E_q(qu_j)u_j^{b_j-1}\,d(q,u_j)\Big\}\frac{1}{2\pi i}\int_C \frac{G(b_1-s)G(b_2-s)G(b_3-\beta_3 s)\pi z^s\prod_{j=1}^{A} \lambda_j^{\alpha_j s}}{G(1-s)\sin\pi s\prod_{j=4}^{B} u_j^{\beta_j s}}\, ds.$$

Similarly on using the known integral due to Hahn [3], (m_1-3)-times and n_1-times, the above expression reduces to

$$\prod_{j=m_1+1}^{B}\Big\{\frac{1}{G(1)2\pi i}\int_{C_j} e_q(u_j)u_j^{-b_j}\,du_j\Big\}\prod_{j=n_1+1}^{A}\Big\{\frac{1}{G(1)2\pi i}\int_{C_j} e_q(\lambda_j)\lambda_j^{-a_j}\,d\lambda_j\Big\}\frac{1}{2\pi i}\int_C \frac{G(b_1-s)G(b_2-s)\prod_{j=3}^{m_1}G(b_j-\beta_j s)\prod_{j=1}^{n_1}G(a_j+\alpha_j s)\prod_{j=n_1+1}^{A}\lambda_j^{\alpha_j s}\,\pi z^s}{G(1-s)\sin\pi s \prod_{j=m_1+1}^{B} u_j^{\beta_j s}}\, ds$$

Now again, changing the order of integration, which is justified as before, and using the known result due to Hahn (3 , §9(b)), $(A-n_1)$-times and $(B-m_1)$-times to evaluate the inner integrals, we finally obtain

$$H^{m_1,n_1}_{A,B}\left[z;q\,\middle|\,\begin{matrix}(1-a_1,\alpha_1),\dots,(1-a_{n_1},\alpha_{n_1}),(a_{n_1+1},\alpha_{n_1+1}),\dots,(a_A,\alpha_A)\\(b_1,1)(b_2,1)(b_3,\beta_3)\dots(b_{m_1},\beta_{m_1})\,(1-b_{m_1+1},\beta_{m_1+1})\dots(1-b_B,\beta_B)\end{matrix}\right] = \frac{1}{2\pi i}\int_C \frac{G(b_1-s)G(b_2-s)\prod_{j=3}^{m_1}G(b_j-\beta_j s)\prod_{j=1}^{n_1}G(a_j+\alpha_j s)\,\pi z^s}{\prod_{j=m_1+1}^{B}G(b_j+\beta_j s)\prod_{j=n_1+1}^{A}G(a_j-\alpha_j s)\,G(1-s)\sin\pi s}\, ds \qquad (5.4)$$

which converges for $|\{\arg z -\omega_2\omega_1^{-1}\log|z|\}| < \pi$.

The results (5.2) and (5.3) can be established in the same way if we employ the results (3.16) and § 9(b) given by Hahn [3], to evaluate the inner integrals.

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