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BASIC STURM - LIOUVILLE THEORY

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ABSTRACT

In the real domain, a basic analogue of a simple form of Sturm-Liouville equation of the second order is studied, and it is shown that, with proper boundary conditions, its solutions are orthogonal with respect to basic integration. Basic functions which are analogous to the sine and cosine are briefly discussed and are utilised in an investigation of the conditions that solutions of the equation under consideration should be oscillatory. Finally, it is shown that an arbitrary function may be expanded in a series of basic eigen-functions. In the limit as q, the base, tends to unity, we recover results which are well-known in ordinary Sturm-Liouville theory.

RESUMEN

En el dominio real, una analogía básica de una forma simple de la ecuación de Sturm-Liouville de segundo orden es estudiada y se muestra

que, con condiciones de contorno apropiadas, estas soluciones son ortogonales en el sentido de la integración básica. Las funciones básicas, las cuales son análogas a seno y coseno, son brevemente discutidas y son utilizadas en una investigación de las condiciones tales que las soluciones de la ecuación sean oscilatorias. Finalmente, se muestra que una función arbitraria puede ser expandida en una serie de funciones propias básicas. En el límite cuando q, la base, tiende a la unidad, obtenemos resultados que son bien conocidos en la teoría ordinaria de Sturm-Liouville.

1. INTRODUCTION

In a long series of papers, F. H. Jackson has studied basic analogues of differentiation and integration and various analogues of special functions. See, for example, [8] and [9]; a complete list of Jackson's papers is given in [1].

Instead of the usual number system, a system of what is referred to as 'basic number' is employed. Such numbers are defined by the relation

$$[a] = (1-q^{a}) / (1-q), \tag{1.1}$$

Where q is any number, real or complex, called the base. It will be seen that, corresponding to the sequence of positive integers 1, 2,...., we have the sequence

$$[1]=1, [2]=1+q, [3]=1+q+q^{2}...,$$

$$[n]=1 + q + q^{2} +... + q^{n-1}$$

In [8], page 255, Jackson introduces the operative symbol Δ defined by

$$\Delta\{\phi(x)\} = \frac{\phi(qx) - \phi(x)}{x(q-1)}, \tag{1.2}$$

Which becomes the same as ordinary differentiation in the limit as q tends to unity, similarly, he defines basic integration as the inverse of basic differentiation, employing the symbol $\overset{b}{\underset{a}{S}}$, which reduces in the limit to $\int_a^b$. These operations correspond exactly, in every way, to differentiation and integration. In order to avoid possible confusion with the usual difference operator Δ, we employ the symbol B_{qx} to mean basic differentiation, and omit the base and independent variable provided that this does not lead to any possibility of misunderstanding.

we have the following elementary results:

$$\hat{B}\, x^n = [n]\, x^{n-1} \qquad \text{and} \qquad \hat{B}\, E_q(ax) = a\, E_q(ax)$$

where $E_q(x) = \sum\limits_{m=o}^{\infty} \frac{x^m}{[m]!}$, $[m]! = [1]\ [2]\ [3] \ldots [m]$,

and corresponds to the exponential function. The basic analogue of the differentiation of a product is used later and is:

$$\hat{B}\{\, u(x)v(x)\} = v(qx)\ \hat{B}\, u\ (x) + u(x)\hat{B}v(x). \tag{1.3}$$

The main purpose of this paper is to establish a basic analogue of the second order Sturm-Liouville system, [6], and to discuss a few of its properties. It is assumed in what follows that all quantities and functions are real unless otherwise stated and that $0 < q \leq 1$.

2. A BASIC ANALOGUE OF THE STURM-LIOUVILLE SYSTEM

Theorem I.

Suppose that the base q is real and such that $0 < q \leq 1$, and that the real functions $r\ (x)$, $l\ (x)$ and $w\ (x)$ possess the appropriate number of q-derivatives on the interval $a \leq x \leq b$, and let $y_m(qx)$ and $y_n\ (qx)$ be eigenfunctions corresponding to distinct eigenvalues λm, λn of the boundary value system

$$\hat{B}\{\ r\ \hat{B}\ y\ (x)\ \} + (l+\lambda w)\ y\ (qx) = 0, \tag{2.1}$$

$$h_1 y + h_2 \hat{B}y = 0 \text{ at } x = a$$

and $\quad k_1 y + k_2 \hat{B}y = 0$ at $x = b$,

h_1, h_2, k_1 and k_2 being constants.

Then $y_m(qx)$ and $y_n(qx)$ are q-orthogonal in the interval $a \leq x \leq b$ with respect to the weight function $w\ (x)$, that is

$$\int_a^b w(x)\, y_m(qx)\, y_n(qx)\, d(qx) = 0, \quad m \neq n. \tag{2.2}$$

<u>Proof.</u> $y_m(x)$ and $y_n(x)$ Satisfy the equations

$$\hat{B}(r\hat{B}y_m) + (1+\lambda_m w)\, y_m(qx) = 0 \tag{2.3}$$

and

$$\hat{B}(r\hat{B}y_n) + (1+\lambda_n w)\, y_n(qx) = 0 \tag{2.4}$$

respectively multiply (2.3) and (2.4) by $y_n(qx)$ and $-\,y_m(qx)$ and add:

$$(\lambda_m-\lambda_n)w(x)y_m(qx)y_n(qx) = y_m(qx)\,\hat{B}\,(r\hat{B}y_n) - y_n(qx)\,\hat{B}\,(r\hat{B}y_m) \tag{2.5}$$

Consider the expression

$$\hat{B}\,\{\, r\hat{B}(y_n).\, y_m - r\hat{B}\,(y_m).y_n \},\tag{2.6}$$

which, on expansion by means of (1.3), becomes

$$\begin{aligned} &y_m(qx)\,\hat{B}\,(r\hat{B}y_n) + r\hat{B}\,(y_n)\,\hat{B}\,(y_m) \\ &- y_n(qx)\,\hat{B}\,(r\hat{B}y_m) - r\hat{B}\,(y_m)\,\hat{B}\,(y_n). \end{aligned}\tag{2.7}$$

This is identical with the right-hand member of (2.5), so that

$$\begin{aligned} &(\lambda_m-\lambda_n)\, w(x)y_m(qx)y_n(qx) = \\ &\hat{B}\,\{\, r\hat{B}(y_n).y_m - r\hat{B}\,(y_m).y_n \}. \end{aligned}\tag{2.8}$$

If we q-integrate with respect to qx between the limits a and b, the result

$$(\lambda_m-\lambda_n)\int_a^b w(x)y_m(qx)y_n(qx)d(qx) = r\hat{B}(y_n).\, y_m - r\hat{B}(y_m).y_n\, \Big|_a^b \tag{2.9}$$

follows immediatelly. The right-hand member of (2.9) is interpreted in the form

$$\begin{aligned} &r(a)\{\hat{B}(y_n)\}_a y_m(a) - r(a)\{\hat{B}(y_m)_a y_n(a) \\ &-r(b)\{\hat{B}(y_n)\}_b y_m(b) + r(b)\{\hat{B}(y_m)_b y_n(b), \end{aligned}\tag{2.10}$$

which clearly vanishes as a consequence of the boundary conditions. Theorem I is thus established.

If $r(a) = 0$ or $r(b) = 0$, then either the first or the second boundary condition respectively may be dropped. In particular, if $r(a)$ and $r(b)$ both vanish, we have the interesting case that the property of q-orthogonality then holds without the imposition of any extrinsic boundary conditions.

Theorem II.

If the basic Sturm-Liouville system of Theorem I satisfies the conditions stated therein, and if the weight function $w(x)$ is either positive throughout the whole interval $a \leq x \leq b$, or negative throughout the same interval, then all the eigenvalues of the system are real.

Proof.

Let $\lambda = \alpha + i\beta$ be an eigenvalue of the problem and suppose that $y(x)=u(x) + iv(x)$ is the corresponding eigenfunction. The quantities α and β and the functions u and v are all real. We then have

$$\hat{B}(r\hat{B}u + ir\hat{B}v) + (1+\alpha w + i\beta w)\left[u(qx) + iv(qx)\right] = 0 \qquad (2.11)$$

This equation is equivalent to the following pair of equations by separating the real and imaginary parts:

$$\hat{B}(r\hat{B}u) + (1+\alpha w)\,u(qx) - \beta wv(qx) = 0 \qquad (2.12)$$

and $$\hat{B}(r\hat{B}v) + (1+\alpha w)\,v(qx) + \beta wu(qx) = 0 \qquad (2.13)$$

Hence, it follows that

$$-\beta\{u(qx)^2 + v(qx)^2\}w = \hat{B}\{r\hat{B}(v).u - r\hat{B}(u).v\}. \qquad (2.14)$$

Again carrying out the process of basic integration between the limits a and b, we find

$$-\beta\int_a^b\{u(qx)^2 + v(qx)^2\}w(x)d(qx) = r\{\hat{B}(v).u - \hat{B}(u).v\}\Big|_a^b \qquad (2.15)$$

Utilising the boundary conditions, we see that the right-hand member of (2.15) vanishes.

However, since $y\ (qx)$ is an eigenfunction, $u(qx)^2 + v(qx)^2 \not\equiv 0$.
Since y and w possess at least two q-derivatives, and w is either greater than or less zero for all x in the interval $a \leq x \leq b$, the q-integral on the left of (2.15) does not vanish. Hence, $\beta = 0$, and $\lambda = \alpha$ and is real. This completes the proof of Theorem II.

3. A BASIC ANALOGUE OF THE SINE AND COSINE

Consider the q-difference equation

$$\hat{B}^2\ y(x) + \lambda^2\ y(qx) = 0, \tag{3.1}$$

which is the simplest special case of (2.1). By a straightforward series development of the solution of this equation, we obtain the two independent solutions $S_q(\lambda x)$ and $C_q(\lambda x)$,
where

$$S_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} x^{2n+1}}{\left[2n+1\right]!} \tag{3.2}$$

and

$$C_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)} x^{2n}}{\left[2n\right]!} \tag{3.3}$$

These two functions reduce respectively to $\sin x$ and $\cos x$ as $q \to 1$.
By means of a numerical investigation they have been shown to be oscillatory for all real values of their arguments. The purpose of mentioning these functions here is that it is necessary to refer to them later in discussing the oscillatory nature of solutions of equation (2.1).
In passing, we note that $S_q(x)$, for example, possesses the orthogonality property

$$\int_0^S \{S_q(\lambda mqx)\ S_q\ (\lambda nqx)\ d\ (qx) = 0, \quad m \neq n; \tag{3.4}$$

S is the first positive zero of $S_q(x)$.
Also, the two functions $S_q(x)$ and $C_q(x)$ are quite distinct from the basic

analogue of the trigonometrical functions which have been studied by previous authors. See [7] and [5] , for instance.

4. A BASIC ANALOGUE OF THE SEPARATION THEOREM

Consider the q-difference equation

$$\hat{B}\{ K \hat{B} y\} - Gy(qx) = 0, \tag{4.1}$$

where the functions K and G are continuous on the close interval $a \leq x \leq b$. By reasoning which is exactly parallel to that employed when discussing the corresponding case for ordinary differential equations, [6] , it follows that (4.1) has only one continuous solution with a continuous basic derivative which satisfies the initial conditions

$$y(c) = \gamma_0, \ (\hat{B}y)_c = \gamma,$$

where c is any point in the closed interval $a \leq x \leq b$. Also, it can be shown that no continuous solution of (4.1) can have an infinite number of zeros in $a \leq x \leq b$ without itself being identically zero. We now proceed to establish the q-analogue of the separation theorem of the zeros of independent solutions of (4.1).

Let y_1 and y_2 be two real, linearly independent solutions of (4.1), such that y_1 vanishes at least twice in the interval $a \leq x \leq b$. If x_1 and x_2 are two consecutive zeros of y_1 in that interval, then it is proposed that y_2 vanishes at least once in the open interval (x_1, x_2). First of all, y_2 cannot vanish at x_1 or at x_2, since it would then not be independent of y_1. Suppose that y_2 does not vanish at any point of (x_1, x_2). The fraction y_1/y_2 is continuous and has a continuous q-derivative in (x_1, x_2). However, we note that

$$\hat{B}(y_1 / y_2) = \frac{y_2 \hat{B} y_1 - y_1 \hat{B} y_2}{y_2 y_2(qx)},$$

and the numerator of this fraction is the basic analogue of the Wronskian of y_1 and y_2, which does not vanish at any point in (x_1, x_2). Hence, this contradiction shows that y_2 must have at least one zero between x_1 and

x_2. There cannot be more than one such zero because if there were two, then y_1 would have a zero between x_1 and x_2 which would not then be consecutives zeros of y_1. Thus the zeros of two real linearly independent solutions of a basic linear difference equation of the second order separate each other.

5. A BASIC ANALOGUE OF STURM'S FUNDAMENTAL OSCILLATION THEOREM

Let u and v, respectively, be solutions of

$$\hat{B}\{\, K \hat{B} u \,\} - G_1 u(qx) = 0 \tag{5.1}$$

and

$$\hat{B}\{\, K \hat{B} v \,\} - G_2 v(qx) = 0, \tag{5.2}$$

where $G_1 \geq G_2$ in $a \leq x \leq b$, but G_1 is not the same as G_2 throughout the whole interval. Multiply (5.1) by v (qx) and (5.2) by u (qx) and subtract, when we have

$$\hat{B}\{\, K \hat{B} u\} v(qx) - \hat{B}\{K\hat{B}v\} u(qx) = (G_1 - G_2) u(qx) v(qx). \tag{5.3}$$

Consider

$$K(v\hat{B}u - u\hat{B}v) = \{\, K\hat{B}u \,\} v - \{K\hat{B}v\} u$$

Expand by means of Jackson's formula (1.3) and obtain

$$\begin{aligned} K(v\hat{B}u - u\hat{B}v) &= v(qx)\hat{B}\{K\hat{B}u\} + \{K\hat{B}u\}\hat{B}v \\ &\quad - u(qx)\hat{B}\{K\hat{B}v\} - \{K\hat{B}v\}\hat{B}u \end{aligned} \tag{5.4}$$

This is the same as the left-hand member of (5.3). Hence, if we q-integrate (5.4) between the limits x_1 and x_2, it follows that

$$\Big[K(v\hat{B}u - u\hat{B}v) \Big]_{x_1}^{x_2} = \int_{x_1}^{x_2} (G_1 - G_2)\, u(qx)\, v(qx)\, d(qx). \tag{5.5}$$

By hypothesis, q is real and positive and such that $0 < q \leq 1$. Let x_1 and x_2 be two consecutive zeros of u and, in the first instance, let v have no zero in (x_1, x_2). Without loss of generality, it may be supposed that both u and v are positive in the interval (x_1, x_2). The right-hand member of (5.4) is then definitely positive. On the left hand side, u is zero at x_1 and x_2, $\hat{B}\, u$ is positive at x_1 and negative at x_2 and v is positive at both limits. The left-hand member of (5.5) is therefore negative and this contradiction shows that v vanishes at least once in (x_1, x_2). In particular, if u and v are both zero at x_1, it is evident that v vanishes before the consecutive zero of u appears. Hence, v oscillates more rapidly than u. A convenient basic analogue of Picone's formula does not appear to exist, so that a more general oscillation theorem has not, so far, been obtained.

We now discuss conditions that the solutions of (4.1) may or may not be oscillatory.
Suppose that, in $a \leq x \leq b$,

$$\underset{\sim}{K} \geqslant K \geqslant \underset{\sim}{k} > 0$$

$$\text{and} \qquad \underset{\sim}{G} \geqslant G \geqslant \underset{\sim}{g}$$

The first comparison equation is

$$\hat{B}\{\, \underset{\sim}{k}\, \hat{B}\, y \,\} - \underset{\sim}{g}\, y(qx) = 0, \tag{5.6}$$

$$\text{or} \qquad \hat{B}^2 y - \underset{\sim}{g}/\underset{\sim}{k}\; y\,(qx) = 0. \tag{5.7}$$

Solutions of (4.1) do not oscillate more rapidly than those of (5.7), which latter is immediately q-integrable.

(i) If $\underset{\sim}{g}=0$, the comparison solution of (5.7) may be taken to be equal to unity, so that, if $\underset{\sim}{g} \geq 0$, the solutions of (5.7) are non-oscillatory for non-positive values of x, that is if $G \geq 0$ in $a \leq x \leq b$, then the solutions of (4.1) are non-oscillatory in that interval provided that no positive value of x is included. This arises because the basic exponential function $E_q(x)$ oscillates for positive real values of x.

(ii) Suppose that $\underset{\sim}{g} < 0$, when we have the oscillatory solution $S_q\{\sqrt{-\underset{\sim}{g}/\underset{\sim}{k}}\, x\}$ of (5.7). Let the positive zeros of $S_q(x)$ be $\alpha_1, \alpha_2 \ldots, \alpha_r, \ldots$ Consecutive zeros of solutions of (5.7) are $\alpha_{r-1}\sqrt{-\underset{\sim}{k}/\underset{\sim}{g}}$ and $\alpha_r\sqrt{-\underset{\sim}{k}/\underset{\sim}{g}}$, so that if $a > \alpha_{r-1}\sqrt{\underset{\sim}{k}/\underset{\sim}{g}}$ and $b < \sqrt{-\underset{\sim}{k}/\underset{\sim}{g}}$, no solutions of (4.1) can have more than one zero in the interval $a \leq x \leq b$.

Consider now the second comparison equation

$$\hat{B}\{\underset{\sim}{K}\hat{B}y\} - \underset{\sim}{G}y(qx) = 0, \tag{5.8}$$

or

$$\hat{B}^2 y - \underset{\sim}{g}/\underset{\sim}{K}\, y(qx) = 0. \tag{5.9}$$

Then the solutions of (4.1) oscillate at least as rapidly as those of (5.9). If G is negative, the solutions of (5.9) are oscillatory and consecutive zeros are $\alpha_{r-1}\sqrt{-\underset{\sim}{K}/\underset{\sim}{g}}$ and $\alpha_r\sqrt{-\underset{\sim}{K}/\underset{\sim}{g}}$. Thus sufficient conditions that the solution of (4.1) should have at least m zeros in the interval $a \leq x \leq b$ are that

$$a < \alpha_{r-m}\sqrt{-\underset{\sim}{K}/\underset{\sim}{G}} \qquad \text{and} \qquad b > \alpha_r\sqrt{-\underset{\sim}{K}/\underset{\sim}{G}}.$$

In particular, sufficient conditions that (4.1) should possess a solution which oscillates in the interval $a \leq x \leq b$ are that

$$a < \alpha_{r-1}\sqrt{-\underset{\sim}{K}/\underset{\sim}{G}} \qquad \text{and} \qquad b > \alpha_r\sqrt{-\underset{\sim}{K}/\underset{\sim}{G}}.$$

6. THE DEVELOPMENT OF AN ARBITRARY FUNCTION IN TERMS OF A SERIES OF q-ORTHOGONAL FUNCTIONS.

We may transform the q-difference equation

$$\hat{B}^2 y + f\hat{B}y + g\,y(qx) = 0 \tag{6.1}$$

by putting $y = u(x)\,v(x)$, and obtain

$$u(q^2x)\,\hat{B}^2v + \{(1+\frac{1}{q})\,\hat{B}u(qx)+fu(qx)\}\,\hat{B}\,v + \{\hat{B}^2u + f\hat{B}u+gu\,(qx)\}v(qx)=0 \quad (6.2)$$

If u is so chosen that

$$(1+\frac{1}{q})\,\hat{B}\,u\,(qx) + fu\,(qx) = 0, \quad (6.3)$$

Then we obtain the normal form of the difference equation (6.1). This is the basic analogue of the process of finding the normal form of the second order linear ordinary differential equation. We may, therefore, without loss of generality, confine our discussion to the normal form of the basic Sturm - Liouville equation (2.1).

Consider the normalised eigenfunctions of the system

$$\begin{aligned} &\hat{B}^2u + \{e^2 - g(x)\}\,u(qx) = 0, \\ &\hat{B}\,u - h\,u = 0 \quad \text{at} \quad x = 0 \\ \text{and} \quad &\hat{B}\,u + H\,u = 0 \quad \text{at} \quad x = 1 \end{aligned} \quad (6.4)$$

in relation to the formal expansion of the arbitrary function f (x) in the form

$$f(x) = \sum_{r-1}^{\infty} u_r\,(qx) \int_0^1 f\,(t)\,u_r\,(qt)\,d\,(qt). \quad (6.5)$$

we recall that our investigation is applied to the real domain only, and that $0 < q \leq 1$.

we may write

$$\int_0^1 f\,(t)\,u_r\,(qt)\,d\,(qt) = -\int_0^1 \frac{f(t)}{e_r^2 - g(t)} \cdot \hat{B}^2\,u_r(t)\,d\,(qt)$$

$$= - \left[\frac{f(t)}{e_r^2 - g(t)} \cdot \hat{B} u_r(t) \right]_0^1 + \int_0^1 \hat{B}\{ \frac{f(t)}{e_r^2 - g(t)} \} . \{ \hat{B} u_r(t) \}_{qt} \, d(qt)$$

$$= - \left[\frac{f(t)}{e_r^2 - g(t)} . \hat{B} u_r(t) + \hat{B}\{ \frac{f(t)}{e_r^2 - g(t)} \} . \frac{1}{q} u_r(qt) \right]_0^1$$

$$- \frac{1}{q} \int_0^1 u_r(q^2 t) \hat{B}^2 \{ \frac{f(t)}{e_r^2 - g(t)} \} \, d(qt)$$

$$= \frac{Hf(1) u_r(1)}{e_r^2 - g(1)} + \frac{hf(0) u_r(0)}{e_r^2 - g(0)} + \frac{1}{q} \left[u_r(qt) \hat{B} \{ \frac{f(t)}{e_r^2 - g(t)} \} \right]_0^1$$

$$- \frac{1}{q} \int_0^1 u_r(qt) \hat{B}^2 \{ \frac{f(t)}{e_r^2 - g(t)} \} \, d(qt). \tag{6.6}$$

This is a consequence of the basic analogue of integration by parts obtained by inverting equation (1.3). If f and g are both continuous and possess continuous first and second q- derivatives, then

$$e^2 \hat{B} \{ \frac{f(t)}{e_r^2 - g(t)} \} \quad \text{and} \quad e^2 \hat{B}^2 \{ \frac{f(t)}{e_r^2 - g(t)} \}$$

are both bounded for sufficiently large values of e and for all t in the interval $0 \leq t \leq 1$.

we see that the series (6.5)converges uniformly in this interval by the M-test.

Let the sum of the series(6.5) be denoted by ψ (x) so that

$$\int_0^1 \Psi(x)\, u_n(qx)\, d(qx) = \sum_{r=1}^{\infty} \int_0^1 u_r(qx) u_n(qx)\, d(qx) \int_0^1 u_r(qt) f(t)\, d(qt)$$

$$= \int_0^1 u_n(qt)\, f(t)\, d(qt), \tag{6.7}$$

because the functions $\{u_r(qt)\}$ constitute a q-orthornormal set. Hence

$$\int_0^1 \{\Psi(x)-f(x)\}\, u_n(qt)\, d(qt)=0 \tag{6.8}$$

for all n, and so $\Psi(x)=f(x)$, and it follows that the expansion (6.5) is convergent in the interval $0 \leq x \leq 1$.

7. CONCLUSION.

The discussion of the basic analogue of the second order Sturm-Liouville system leads potentially to several new classes of functions which are orthogonal with respect to basic integration. A basic analogue of certain properties of the oscillation of solutions of the associated q-difference equation is also given. Finally, it is shown that a wide class of functions may be expanded in series of basic eigenfunctions.

Although the equation of q-orthogonality has been discussed from a different point of view by other authors, see [4], for example, any relations which may exist between the results implicit here and those obtained elsewhere remain to be investigated. It would appear that the interesting posibility of introducing a number of new analogues of the classical orthogonal functions arises from the general result proved in the second section of this paper. A few of the properties of a basic Laguerre polynomial which satisfies a q-difference equation of the type (2.1) have been investigated by the author in [2] and [3] and it is hoped to develope this matter at greater lenght subsequently. All the results given here reduce to the corresponding results in ordinary analysis when $q \to 1$.

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