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GENERAL EXPECTATION OF PARTITIONAL MOMENT FUNCTIONS

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ABSTRACT

The paper provides a general theory for expected values of linear functions of products of sample power sums in terms of products of population power sums - all given symbolically by partitions. This approach is so general that the results can be applied to any sample moment function under any sampling law from a finite or infinite, univariate or multivariate, population. With simple modification, an unbiased estimate of the population moment function in the above situations can also be determined. The results provided are general enough to cover most of the work done so far on moments of moments. The results feature coefficients of individual terms, thereby avoiding accumulated algebraic errors, frequent in earlier works.

RESUMEN

Este trabajo presenta una teoría general para valores esperados de funciones lineales de productos de "sample power sums" en términos de productos de "population power sums", todas dadas simbólicamente por particiones. Esta formulación es tan general que los resultados pueden ser aplicados a cualquier "sample moment function" bajo cualquier ley de muestreo, para poblaciones finitas o infinitas, univariadas o multivariadas. Con una simple modificación también se puede obtener, para las situaciones antes mencionadas, un estimado no sesgado de la "population moment function".

Los resultados presentados son lo suficientemente generales como para cubrir la mayoría de los trabajos realizados hasta el momento sobre momentos de momentos.

Los resultados también presentan coeficientes de términos individuales, eliminando de esta manera la acumulación de errores algebraicos tan frecuentes en trabajos anteriores.

1. INTRODUCTION

The purpose of this paper is to present a most inclusive theory for the expected value of sample moment functions by identifying the resulting formulae, essentially, by partitions. The theory is inclusive enough to cover

a) different linear functions of the products of sample power sums - thus treating k-statistics, sample central moments, and other sample moment functions at the same time;
b) sampling from a finite universe as well as from an infinite supply - the moment laws for infinite sampling are identical with those of finite sampling with replacements;
c) different replacement laws - much more general than those usually considered;
d) multivariate as well as univariate populations - the basic treatment uses a column for each unit variable, so univariate and other multivariate results come from combining (coalescing) columns;
e) populations with different moment characterizations - the results are in terms of power sums and so are applicable to all distributions which are characterized by their moments (power sums);
f) the formula for an unbiased estimate of any linear function of products of population power sums subject to (b), (c), (d), (e) - included are population central moments, cumulants, products of cumulants, etc.

There appears to be little in the literature which can match this generality, though estimates of cumulants and of products of cumulants have previously been used in place of (a) to avoid the complexity of the conventional results. With the approach of this paper, all the cases above are covered by one general result. Almost any result obtained during the last century on the expectation of sample moment functions, besides all the new ones, can be obtained by specifying the values in (a) - (e). Because the elements of the formula are identified by partitions , the coefficient of a particular moment product, in the result, can be obtained if desired.

The presentation here deals with expectation , and unbiased estimation, only but other moment functions of the sample moments can be obtained therefrom by conventional formulae, and more easily with the use of partitions.

A natural expressions of the central moments in terms of products of power sums features partitions, so it is not surprising that formulae for moments of sample moments have been organized around the resulting partitions . This was first demonstrated successfully by Fisher [7] who , by chosing the sample function to be an estimate of a cumulant , was able to introduce so much simplicity in the result that , with a few additional facts such as the determination of the algebraic coefficient of the partition , the sets of partitions serve as the formulae (for sampling from an infinite univariate population).

This idea of using the partitions themselves as the components of the solutions was extended by Dwyer [3] to a general sample function . Results were obtained for expectations of moment functions which are general linear combinations of products of power sums, and from these, to determine formulae for moment functions of the sample moment functions , again for sampling from an infinite univariate population.

Since Fisher had such success in changing the problem from sample moments to k-statistics , Tukey [11] attacked the more general finite problem (sampling without replacement) by using polykays as the sample , function having the property that the expected value is a product of cumulants . The theory was further developed by Wishart [12] and Kendall [8] , and the results related to the partitions of multipartite numbers . The polykays do have the nice property , by definition , that they are unbiased estimates of products of cumulants, but, for higher orders, they differ appreciably from the simple sample moment functions . Also this nice property is not readily extendable to an algebra using them as it does not extend , without extensive work, even to the product of two of them [5, p.41].

There is another approach [1], [4], giving results for finite populations, which does not require that the problem be changed so as to get simpler results identified by partitions. This approach, partially generalized in [5], is given more extensive generalization here for expectation, and new general results for unbiased estimation are presented.

2. NOTATION, DEFINITIONS AND SOME PREVIOUS RESULTS

We summarize, briefly, the concepts, notation and basic facts required. These are generally quite consistent with those of [1] - [6], [9], [10].

a. Partitions. A most important concept for this paper is that of a general partition. We consider multipartite number $u_r = \underbrace{11 \ldots 1}_{r}$ consisting of r units. The partitions of u_r are formed by placing the units in different rows, with at least one unit in each row (the remaining spaces are filled with zeros). Each partition is unique, i.e., it can arise by partitioning u_r in only one way (a permutation of its rows does not alter the partition). The rows represent the parts of the partition, and the columns represent different variables. When certain variables are identical, the corresponding columns are combined (coalesced) by adding the corresponding elements in the rows. When all variables are the same, all columns are coalesced, yielding column vector $\underline{r}$, whose elements are the parts of unipartite r. The number of partitions of the multipartite number u_r which coalesce to $\underline{r}$ is called $\phi(r)$, the combinatorial coefficient of $\underline{r}$. If $\underline{r}$ contains π parts, with π_i r_i's, $r_1 > r_2 > \ldots > r_h$, $r = \sum_1^h r_i \pi_i$, $\pi = \sum_1^h \pi_i$ and

$$\phi(\underline{r}) = \frac{r!}{(r_1!)^{\pi_1} \ldots (r_h!)^{\pi_h} \pi_1! \ldots \pi_h!} \tag{2.1}$$

since an interchange of equal rows does not change $\underline{\pi}$.

For a multipartite partition R, the combinatorial coefficient $\phi(R)$ is the product of the combinatorial coefficients of the individual columns , except that the $\pi_1! \ldots \pi_h!$ term is applied to the repeated rows (parts) of R, rather than to the repeated entries of the individual columns.

Thus

$$\phi\begin{pmatrix}2\\2\\2\\2\end{pmatrix} = \frac{8!}{(2!)^4\, 4!} = 105, \text{ while } \phi\begin{pmatrix}2 & 0\\0 & 2\\1 & 1\\1 & 1\end{pmatrix} = \frac{4!}{2!(1!)^2} \cdot \frac{4!}{2!(1!)^2} \cdot \frac{1}{2!} = 72 .$$

Also partitions themselves may be partioned. Thus $\begin{matrix}110\\001\end{matrix}$ has partitions $\begin{matrix}110\\001\end{matrix}$ and $\begin{matrix}100\\010\\001\end{matrix}$.

b. Power sums and power product sums. Let x_α be the α^{th} member of a sample of size n or of a finite population of size N. Then power sums for the univariate case are

$$\sum_1^n x_\alpha^g = (g), \quad \sum_1^N x_\alpha^g = (g)_N$$

and for the multivariate case

$$\sum_1^n x_{1\alpha}^{g_1} x_{2\alpha}^{g_2} \ldots x_{r\alpha}^{g_r} = (g_1\, g_2\, \ldots\, g_r) \quad \text{for the sample, and}$$

$$\sum_1^N x_{1\alpha}^{g_1} x_{2\alpha}^{g_2} \ldots x_{r\alpha}^{g_r} = (g_1\, g_2\, \ldots\, g_r)_N \quad \text{for the population.}$$

In what follows , it is frequently required to use products of power sums for the parts of a partition. The notation used is to enclose the partition in parentheses. Thus $\begin{pmatrix} p_1 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}$ is a symbol for the $(p_1\ p_2\ 0)(0\ 0\ p_3)$.

Similarly $\sum_{\alpha \neq \beta} x_\alpha^g x_\beta^h$ is indicated by $[g\ h]$ for the sample, and by $[g\ h]_N$ for the population, and called a power product sum [4, p.13] or augmented mononial symmetric function [2, p.2].

Relations expressing products of power sums in terms of power product sums, and vice versa, are important in this direct general sampling theory. The basic multiplication theorem for power sums [4, p.15] is given by

$$(Q) = \sum_W [W] \tag{2.2}$$

where W is any partition which results from adding parts of Q, both being partitions of u_r. This, in a sense, generalizes

$$\begin{pmatrix} a \\ b \end{pmatrix} = [a+b] + \begin{bmatrix} a \\ b \end{bmatrix}$$

or

$$\sum x_\alpha^a \sum x_\alpha^b = \sum x_\alpha^{a+b} + \sum_{\alpha \neq \beta} x_\alpha^a x_\beta^b .$$

We also need to determine $[W]$ in terms of (R), where R is any partition resulting from coalescing parts of W. If W has s parts, R has t parts, and $s_1, s_2, \ldots, s_t$ are the numbers of rows (parts) of W coalesced to form the successive rows of R, $(s = \sum s_i)$, then [4, p.30]

$$[W] = \sum_R (-1)^{s-t} (s_1 - 1)!(s_2 - 1)! \dots (s_t - 1)! \; (R) . \qquad (2.3)$$

We need a formula for the expected value of W. A general formula is [10, p.13]

$$E[W] = d_W [W]_N \qquad (2.4)$$

where d_W is a function of W. For sampling without replacement, $d_W = e_s = n^{(s)}/N^{(s)}$ where s is the number of parts of W. For sampling with replacement, $d_W = n^{(s)}/N^s$. For many sampling laws, d_W depends only on the number of parts of W, and hence can be represented by d_s.

c. Symmetric means. Also called angle bracket [11], it is defined as $\langle g \rangle = \dfrac{[g]}{n^{(\pi)}}$ for the sample, and $\langle g \rangle_N = \dfrac{[g]_N}{N^{(\pi)}}$ for the population. Letting E_N denote expectation when sampling without replacement from a finite population

$$E_N \langle g \rangle = \langle g \rangle_N \qquad (2.5)$$

Thus $\langle g \rangle$ provides an unbiased estimate of $\langle g \rangle_N$, a property much used for k-statistics [11] and h-statistics [3, p.26].

3. MOMENT FUNCTIONS AS PARTITIONAL FUNCTIONS

Dwyer [3, p.23] defined a general moment function δ_p in terms of power sums of partitions of p. Here we first define a function δ_{u_r} which is a linear combination of the partitional power sums of u_r. Thus

$$\delta_{u_r} = \sum_Q a_Q (Q) \tag{3.1}$$

where Q is any partition of u_r and (Q) is the product of the power sums of its rows. Thus

$$\delta_{u_1} = a_1 (1)$$

$$\delta_{u_2} = a_{11}(11) + a_{\substack{10\\01}} \binom{10}{01}$$

$$\delta_{u_3} = a_{111}(111) + a_{\substack{110\\001}} \begin{pmatrix}110\\001\end{pmatrix} + a_{\substack{101\\010}} \begin{pmatrix}101\\010\end{pmatrix} + a_{\substack{011\\100}} \begin{pmatrix}011\\100\end{pmatrix} = a_{\substack{100\\010\\001}} \begin{pmatrix}100\\010\\001\end{pmatrix}.$$

Values of a's remain unchanged on interchanging or coalescing columns. However, non-unit combinatorial coefficients (Sec.2) begin to show up on such coalescing.

In notation similar to (3.1), we have for populations,

$$F_{u_r} = \sum_Q A_Q (Q)_N \tag{3.2}$$

where F and A indicate corresponding population functions in place of sample function.

4. EXPECTATION OF SAMPLE PARTITIONAL FUNCTIONS

Application of the multiplication theorem (2.2) for power sums to the definition of $s_{u_r} = \sum_Q a_Q \, (Q)$ in (3.1) gives at once

$$s_{u_r} = \sum_Q a_Q \sum_W \, [W] \ .$$

For expectation, using (2.4) we have

$$E\left(s_{u_r}\right) = \sum_Q a_Q \sum_W d_W \, [W]_N \ .$$

Expanding W_N in terms of power sums $(R)_N$ by (2.3), and collecting the coefficients of $(R)_N$,

$$E\left(s_{u_r}\right) = \sum_R \sum_Q a_Q \sum_W d_W \, (-1)^{s-t} \, (s_1 - 1)! \ldots (s_t - 1)! \, (R)_N$$

$$= \sum_R \sum_Q a_Q \, c_{Q|R} \, (R)_N$$

$$= \sum_R v_R \, (R)_N \ . \qquad (4.1)$$

where

$$C_{Q|R} = \sum_W d_W \, (-1)^{s-t} \, (s_1 - 1)! \ldots (s_t - 1)! \,, \qquad (4.2)$$

$$D_R = \sum_Q a_Q \, C_{Q|R} \,. \qquad (4.3)$$

It is here seen that partitions R represent all the terms of $E\left(\delta_{u_r}\right)$, and that the coefficients D_R are also represented by these partitions. These relations are illustrated in Table 1 where the terms of $E\left(\delta_{u_r}\right)$ are all presented explicitly for $r = 3$. The values of $C_{Q|R}$ (which become C_π, with column vector $\underline{\pi}$, when $d_W = d_s$ as explained below) are indicated in the interior of the table. The columns are multiplied by the a_Q in the left margin and the sums formed to obtain the D_R placed in the bottom row. These D_R's are multiplied by the $(R)_N'$s of the top row and added to obtain $E\left(\delta_{u_3}\right)$.

When R is a one-part partition, e.g. 111 in Table 1, $t = 1$ and $s_1 = s$, so (4.1) becomes

$$C_{Q|R} = \sum_W (-1)^{s-1} \, (s - 1)! \, d_s \qquad (4.4)$$

which we call the generalized Carver function C_r. Special cases are $C_1 = d_1$, $C_2 = d_1 - d_2$, $C_3 = d_1 - 3d_2 + 2d_3$.

TABLE 1

VALUES OF $C_{Q|R}$ AND D_R IN $E\left(\delta_{u_3}\right)$

Q	a_Q	R / $E(Q) \diagdown (R)_N$	111	110 001	101 010	011 100	100 010 001
			$(111)_N$	$\begin{pmatrix}110\\001\end{pmatrix}_N$	$\begin{pmatrix}101\\010\end{pmatrix}_N$	$\begin{pmatrix}011\\100\end{pmatrix}_N$	$\begin{pmatrix}100\\010\\001\end{pmatrix}_N$
111	a_{111}	$E(111)$	$d_1 = C_1$				
110 001	$a_{\substack{110\\001}}$	$E\begin{pmatrix}110\\001\end{pmatrix}$	$d_1-d_2 = C_2$	$d_2 = C_{\substack{1\\1}}$			
101 010	$a_{\substack{101\\010}}$	$E\begin{pmatrix}101\\010\end{pmatrix}$	$d_1-d_2 = C_2$		$d_2 = C_{\substack{1\\1}}$		
011 100	$a_{\substack{011\\100}}$	$E\begin{pmatrix}011\\100\end{pmatrix}$	$d_1-d_2 = C_2$			$d_2 = C_{\substack{1\\1}}$	
100 010 001	$a_{\substack{100\\010\\001}}$	$E\begin{pmatrix}100\\010\\001\end{pmatrix}$	$d_1-3d_2+2d_3=C_3$	$d_2-d_3=C_{\substack{2\\1}}$	$d_2-d_3=C_{\substack{2\\1}}$	$d_2-d_3=C_{\substack{2\\1}}$	$d_3 = C_{\substack{1\\1\\1}}$
		$E\left(\delta_{u_3}\right)$	D_{111}	$D_{\substack{110\\001}}$	$D_{\substack{101\\010}}$	$D_{\substack{011\\100}}$	$D_{\substack{100\\010\\001}}$

When R is a two-part partition, with r_1 units in the first row and r_2 in the second, applying (4.3) to the s_1 rows of W which coalesce to the first row of R, and again to the s_2 rows of W which coalesce to the second row of R, we get

$$C_{Q|R} = \sum_W (-1)^{s_1 - 1} (s_1 - 1)!\, d_{s_1} (-1)^{s_2 - 1} (s_2 - 1)!\, d_{s_2}$$

$$= \sum_W (-1)^{s-2} (s_1 - 1)!\, (s_2 - 1)!\, d_s$$

$$= C_{r_1} \circ C_{r_2} ,$$

where $\circ$ indicates the addition of the subscripts of d's (i.e. the number of parts) . This is denoted by $C_{\substack{r_1 \\ r_2}}$ in Table 1 for the 2-part partitions R. Here,

$$C_{\substack{1 \\ 1}} = C_1 \circ C_1 = d_1 \circ d_1 = d_2$$

$$C_{\substack{2 \\ 1}} = C_2 \circ C_1 = (d_1 - d_2) \circ d_1 = d_2 - d_3 .$$

The argument is immediately extended for $t > 2$, and with π denoting the column vector $(r_1, r_2, \dots, r_t)'$, $C_\pi = C_{r_1} \circ C_{r_2} \circ \dots \circ C_{r_t}$.

Thus, in Table 1, $C_{\substack{1 \\ 1 \\ 1}} = d_1 \circ d_1 \circ d_1 = d_3$.

For all sampling laws , where $d_W = d_s$, depending only on the number of parts of W, the C's for univariate or multivariate cases are identical, being dependent only on the number of parts . By coalescing the columns in Table 1, we obtain Table 2 for $E(\delta_3)$, which displays non-unit combinatorial coefficients (2.1).

TABLE 2

VALUES OF $C_{Q|R}$ AND D_R IN $E(\sigma_3)$

		R	3	$\begin{matrix}2\\1\end{matrix}$	$\begin{matrix}1\\1\\1\end{matrix}$
Q	a_Q	$(R)_N$ \ $E(Q)$	$(3)_N$	$\begin{pmatrix}2\\1\end{pmatrix}_N$	$\begin{pmatrix}1\\1\\1\end{pmatrix}_N$
3	a_3	$E(3)$	C_1		
$\begin{matrix}2\\1\end{matrix}$	$a_{\substack{2\\1}}$	$E\begin{pmatrix}2\\1\end{pmatrix}$	C_2	$C_{\substack{1\\1}}$	
$\begin{matrix}1\\1\\1\end{matrix}$	$a_{\substack{1\\1\\1}}$	$E\begin{pmatrix}1\\1\\1\end{pmatrix}$	C_3	$3C_{\substack{2\\1}}$	$C_{\substack{1\\1\\1}}$
		$E(\sigma_3)$	D_3	$D_{\substack{2\\1}}$	$D_{\substack{1\\1\\1}}$
		Combl.coeff.	1	3	1

If only the first two columns of R are coalesced in Table 1, we obtain the case of $E(\sigma_{21})$, as in Table 3.

TABLE 3

VALUES OF $C_{Q|R}$ AND $\mathcal{D}_R$ IN $E(\delta_{21})$

		R	21	$\begin{matrix}20\\01\end{matrix}$	$\begin{matrix}11\\10\end{matrix}$	$\begin{matrix}10\\10\\01\end{matrix}$
Q	a_Q	$E(Q)$ \ $(R)_N$	$(21)_N$	$\begin{pmatrix}20\\01\end{pmatrix}_N$	$\begin{pmatrix}11\\10\end{pmatrix}_N$	$\begin{pmatrix}10\\10\\01\end{pmatrix}_N$
21	a_{21}	$E(21)$	C_1			
$\begin{matrix}20\\01\end{matrix}$	$a_{\substack{20\\01}}$	$E\begin{pmatrix}20\\01\end{pmatrix}$	C_2	$C_{\substack{1\\1}}$		
$\begin{matrix}11\\10\end{matrix}$	$a_{\substack{11\\10}}$	$E\begin{pmatrix}11\\10\end{pmatrix}$	C_2		$C_{\substack{1\\1}}$	
$\begin{matrix}10\\10\\01\end{matrix}$	$a_{\substack{10\\10\\01}}$	$E\begin{pmatrix}10\\10\\01\end{pmatrix}$	C_3	$C_{\substack{2\\1}}$	$2C_{\substack{2\\1}}$	$C_{\substack{1\\1\\1}}$
		$E(\delta_{21})$	$\mathcal{D}_{21}$	$\mathcal{D}_{\substack{20\\01}}$	$\mathcal{D}_{\substack{11\\10}}$	$\mathcal{D}_{\substack{10\\10\\01}}$
		Combl.Coeff.	1	1	2	1

We also present Table 4 below for $E(\delta_4)$.

TABLE 4

VALUES OF $C_{Q|R}$ AND D_R IN $E(\delta_4)$

		R	4	$\begin{matrix}3\\1\end{matrix}$	$\begin{matrix}2\\2\end{matrix}$	$\begin{matrix}2\\1\\1\end{matrix}$	$\begin{matrix}1\\1\\1\\1\end{matrix}$
Q	a_Q	$E(Q)$ \ $(R)_N$	$(4)_N$	$\begin{pmatrix}3\\1\end{pmatrix}_N$	$\begin{pmatrix}2\\2\end{pmatrix}_N$	$\begin{pmatrix}2\\1\\1\end{pmatrix}_N$	$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}_N$
4	a_4	$E(4)$	C_1				
$\begin{matrix}3\\1\end{matrix}$	$a_{\substack{3\\1}}$	$E\begin{pmatrix}3\\1\end{pmatrix}$	C_2	$C_{\substack{1\\1}}$			
$\begin{matrix}2\\2\end{matrix}$	$a_{\substack{2\\2}}$	$E\begin{pmatrix}2\\2\end{pmatrix}$	C_2		$C_{\substack{1\\1}}$		
$\begin{matrix}2\\1\\1\end{matrix}$	$a_{\substack{2\\1\\1}}$	$E\begin{pmatrix}2\\1\\1\end{pmatrix}$	C_3	$2C_{\substack{2\\1}}$	$C_{\substack{2\\1}}$	$C_{\substack{1\\1\\1}}$	
$\begin{matrix}1\\1\\1\\1\end{matrix}$	$a_{\substack{1\\1\\1\\1}}$	$E\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$	C_4	$4C_{\substack{3\\1}}$	$3C_{\substack{2\\2}}$	$6C_{\substack{2\\1\\1}}$	$C_{\substack{1\\1\\1\\1}}$
		$E(\delta_4)$	D_4	$D_{\substack{3\\1}}$	$D_{\substack{2\\2}}$	$D_{\substack{2\\1\\1}}$	$D_{\substack{1\\1\\1\\1}}$
		Combl. Coeff.	1	4	3	6	1

5. UNBIASED ESTIMATION

Since $E[W] = d_W [W]_N$, with $d_W > 0$, it follows that the unbiased estimate of $[W]_N$ is

$$E^{-1}\ [W]_N = \frac{1}{d_W}[W] = d_W^* \ [W].$$

Then, as in Section 3, with

$$F_{u_r} = \sum_Q A_Q \ (Q)_N = \sum_Q A_Q \sum_W \ [W]_N \ ;$$

$$E^{-1}\left(F_{u_r}\right) = \sum_Q A_Q \sum_W d_W^* \ [W]$$

$$= \sum_R \sum_Q A_Q \sum_W d_W^* \ (-1)^{s-t} \ (s_1-1)! \ldots (s_t-1)! \ (R)$$

$$= \sum_R \sum_Q A_Q \ C_{Q|R}^* \ (R)$$

$$= \sum_R D_R^* \ (R)$$

where $D_R^* = \sum_Q A_Q \ C_{Q|R}^*$ and $C_{Q|R}^*$ is $C_{Q|R}$ with d_W^* replacing d_W. When $d_W = d_s$, $C_{Q|R}^*$ becomes C_r^*, which is C_r with $\frac{1}{d_s}$ replacing d_s. For sampling without replacement, $d_s = e_s = \frac{n^{(s)}}{N^{(s)}}$ so then C_r^* are Carver functions with n, N interchanged.

Again we see that the results for unbiased estimation are given symbolically by the partitions of R and Q. Also, univariate and other multivariate results are obtained by coalescing columns.

6. ALTERNATIVE FORMULAE USING SYMMETRIC MEANS

For some purposes it may be useful to express δ_{u_r} in terms of symmetric means [11] $\langle W\rangle = \frac{1}{n^{(s)}}[W]$. For example, the definitions of k-statistics and h-statistics are naturally in terms of symmetric means. This representation is fine a) if the δ_{u_r} can be represented naturally as a linear function of symmetric means, or b) if one ignores the difficulties in transforming products of power sums to symmetric means, or vice versa. Though, for sampling without replacement, $E\langle W\rangle = \langle W\rangle_N$ (2.5), it does not follow that $E\langle W_1\rangle\langle W_2\rangle$ is easy, since $\langle W_1\rangle\langle W_2\rangle$ is not transformed easily to a linear function of $\langle W\rangle$'s. See [5, p.41].

However, for any δ_{u_r} which is readily expressible as a linear function of symmetric means, the formulation $\delta_{u_r} = \sum_W b_W \langle W\rangle$ is very useful since, at once, $E(\delta_{u_r}) = \sum_W b_W \langle W\rangle_N$. Expressing this result in terms of products of power sums,

$$E(\delta_{u_r}) = \sum_W \frac{b_W}{N^{(s)}} [W]_N$$

$$= \sum_R \sum_W \frac{b_W}{N^{(s)}} (-1)^{s-t} (s_1-1)! \dots (s_t-1)! \, (R)_N$$

which is again of the form $\sum_R v_R (R)_N$.

7. PRODUCTS OF PARTITIONAL FUNCTIONS

We have shown above how to obtain the expectation of a partitional function. Our next concern is the expectation of products of partitional functions, needed right away if we want to investigate their moments, product moments, cumulants, etc. All that is required here is a consideration of a partitional function whose weight equals the weight of the product. Thus $\delta_{10} = a_1(10)$, $\delta_{01} = a_1(01)$ and the product $\delta_{10}\delta_{01} = a_1^2 \begin{pmatrix} 10 \\ 01 \end{pmatrix}$, which is

$$\delta_{11} = a_{11}(11) + a_{\substack{10\\01}}\begin{pmatrix} 10 \\ 01 \end{pmatrix}, \text{ with } a_{11} = 0,\ a_{\substack{10\\01}} = a_1^2 .$$

Using $E_N(\delta_{11}) = \mathcal{D}_{11}\ (11)_N + \mathcal{D}_{\substack{10\\01}} \begin{pmatrix} 10 \\ 01 \end{pmatrix}_N$

$$= (a_{11}\, C_1 + a_{\substack{10\\01}}\, C_2)(11)_N + a_{\substack{10\\01}}\, C_{\substack{1\\1}} \begin{pmatrix} 10 \\ 01 \end{pmatrix}_N ,$$

we obtain $E_N(\delta_{10}\delta_{01}) = a_1^2\, C_2\ (11)_N + a_1^2\, C_{\substack{1\\1}} \begin{pmatrix} 10 \\ 01 \end{pmatrix}_N$

In general, the product of two δ's can be written

$$\delta_{u_{r_1}} \delta_{u_{r_2}} = \Sigma_{Q_1}\, a_{Q_1}\, (Q_1) \cdot \Sigma_{Q_2}\, a_{Q_2}\, (Q_2) \tag{7.1}$$

$$= \Sigma_Q\, a_Q\, (Q) \tag{7.2}$$

where $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, where the first r_1 and the last r_2 columns of

Q are reserved for Q_1, Q_2 respectively, and $a_Q = a_{Q_1} a_{Q_2}$. Thus

$$\delta_{11}\,\delta_1 = \left\{ a_{11}(11) + a_{\substack{10\\01}} \begin{pmatrix}10\\01\end{pmatrix} \right\} a_1(1)$$

can be written as

$$\delta_{110}\delta_{001} = \left\{ a_{110}(110) + a_{\substack{100\\010}} \begin{pmatrix}100\\010\end{pmatrix} \right\} a_{001}(001)$$

$$= a_{\substack{110\\ \dot{0}\dot{0}\dot{1}}} \begin{pmatrix}110\\001\end{pmatrix} + a_{\substack{100\\010\\ \dot{0}\dot{0}\dot{1}}} \begin{pmatrix}100\\010\\001\end{pmatrix} \tag{7.3}$$

where the dotted lines distinguish the subscripts of the a's. Then, applying (4.2),

$$E\left|\delta_{110}\ \delta_{001}\right| = \mathcal{D}_{111}(111)_N + \mathcal{D}_{\substack{110\\001}} \begin{pmatrix}110\\001\end{pmatrix}_N + \mathcal{D}_{\substack{101\\010}} \begin{pmatrix}101\\010\end{pmatrix}_N + \mathcal{D}_{\substack{011\\100}} \begin{pmatrix}011\\100\end{pmatrix}_N$$

$$+ \mathcal{D}_{\substack{100\\010\\001}} \begin{pmatrix}100\\010\\001\end{pmatrix}_N \tag{7.4}$$

with $\mathcal{D}_{111} = a_{\substack{110\\ \dot{0}\dot{0}\dot{1}}} C_2 + a_{\substack{100\\010\\ \dot{0}\dot{0}\dot{1}}} C_3$, $\mathcal{D}_{\substack{110\\001}} = a_{\substack{110\\ \dot{0}\dot{0}\dot{1}}} C_{\substack{1\\1}} + a_{\substack{100\\010\\ \dot{0}\dot{0}\dot{1}}} C_{\substack{2\\1}}$,

$\mathcal{D}_{\substack{101\\010}} = a_{\substack{100\\010\\ \dot{0}\dot{0}\dot{1}}} C_{\substack{2\\1}}$, $\mathcal{D}_{\substack{011\\100}} = a_{\substack{100\\010\\ \dot{0}\dot{0}\dot{1}}} C_{\substack{2\\1}}$, $\mathcal{D}_{\substack{100\\010\\001}} = a_{\substack{100\\010\\ \dot{0}\dot{0}\dot{1}}} C_{\substack{1\\1\\1}}$.

It may be noted that for each R, the coefficient can be determined without reference to other R's, from the values of the Q's in (7.3) which coalesce, by rows, to it.

Formula (7.4) can take special cases. Thus for moments,

$m_{110} = \delta_{110}$ with $a_{110} = \frac{1}{n}$, $a_{\substack{100\\010}} = -\frac{1}{n^2}$, and $m'_{001} = \delta_{001}$ with $a_{001} = \frac{1}{n}$.

One thus obtains

$$E\left(m_{110}m'_{001}\right) = \frac{1}{n^3}\left\{(n-1)e_1 - (n-3)e_2 - 2e_3\right\}NM_{111}$$

and by coalescing further

$$E\left(m_{20}m'_{01}\right) = \frac{1}{n^3}\left\{(n-1)e_1 - (n-3)e_2 - 2e_3\right\}NM_{21}$$

$$E\left(m_{11}m'_{01}\right) = \frac{1}{n^3}\left\{(n-1)e_1 - (n-3)e_2 - 2e_3\right\}NM_{12}$$

$$E\left(m_2 m'_1\right) = \frac{1}{n^3}\left\{(n-1)e_1 - (n-3)e_2 - 2e_3\right\}NM_3 .$$

The results can be extended to products of more than two δ's .

8. CENTRAL POPULATION PARTITIONAL FUNCTIONS

For many purposes it is satisfactory to take the origin at the population mean. In expectation problems there is no essential loss in generality if the population mean is also known and for central moments, even this is not necessary. Population power sums of deviates are here denoted by $(\ )_{N.}$.

This specification indicates that the expectation formulae above for central partitional functions require only those values of $(R)_{N.}$ which have no unit parts, so the corresponding $\mathcal{D}_R$'s need not be computed. This is specially useful for large r; even for $r = 4$, only 4 of the 15 R's, viz. 1111, $\begin{matrix}1100\\0011\end{matrix}$, $\begin{matrix}1010\\0101\end{matrix}$, $\begin{matrix}1001\\0110\end{matrix}$, have non-vanishing coefficients. Thus in forming $E(\delta_{1100}\delta_{0011})$ where $\delta_{1100} = a_{1100}(1100) + a_{\substack{1000\\0100}}\begin{pmatrix}1000\\0100\end{pmatrix}$ and $\delta_{0011} = a_{0011}(0011) + a_{\substack{0010\\0001}}\begin{pmatrix}0010\\0001\end{pmatrix}$, we need consider the contributions only to $(1111)_{N.}$, $\begin{pmatrix}1100\\0011\end{pmatrix}_{N.}$, $\begin{pmatrix}1010\\0101\end{pmatrix}_{N.}$ and $\begin{pmatrix}1001\\0110\end{pmatrix}_{N.}$ by the Q's listed in Table 5. The last column in the table is used to record the products of the a's.

TABLE 5

VALUES OF $\mathcal{D}_R$ IN $E\left(\sigma_{1100}\ \sigma_{0011}\right)$ WITH POPULATION MEAN 0

$Q \diagdown (R)_{N.}$	$(1111)_{N.}$	$\begin{pmatrix}1100\\0011\end{pmatrix}_{N.}$	$\begin{pmatrix}1010\\0101\end{pmatrix}_{N.}$	$\begin{pmatrix}1001\\0110\end{pmatrix}_{N.}$	a_Q
1100 0011	C_2	$C_{\substack{1\\1}}$			$a_{1100} \times a_{0011}$
1100 .... 0010 0001	C_3	$C_{\substack{2\\1}}$			$a_{1100} \times a_{\substack{0010\\0001}}$
0011 .... 1000 0100	C_3	$C_{\substack{2\\1}}$			$a_{0011} \times a_{\substack{1000\\0100}}$
1000 0100 .... 0010 0001	C_4	$C_{\substack{2\\2}}$	$C_{\substack{2\\2}}$	$C_{\substack{2\\2}}$	$a_{\substack{1000\\0100}} \times a_{\substack{0010\\0001}}$
$E\left(\sigma_{1100}\ \sigma_{0011}\right)$	$\mathcal{D}_{1111}$	$\mathcal{D}_{\substack{1100\\0011}}$	$\mathcal{D}_{\substack{1010\\0101}}$	$\mathcal{D}_{\substack{1001\\0110}}$	

Hence, the covariance

$$M_{11}\left(\sigma_{1100}, \sigma_{0011}\right) = E\left(\sigma_{1100}\ \sigma_{0011}\right) - E\left(\sigma_{1100}\ \ E\,\sigma_{0011}\right)$$

$$= \mathcal{D}_{1111}(1111)_{N.} + \left(\mathcal{D}_{\substack{1100\\0011}} - \mathcal{D}_{1100}\ \mathcal{D}_{0011}\right)\begin{pmatrix}1100\\0011\end{pmatrix}_{N.}$$

$$+ \mathcal{D}_{\substack{1010\\0101}}\begin{pmatrix}1010\\0101\end{pmatrix}_{N.} + \mathcal{D}_{\substack{1001\\0110}}\begin{pmatrix}1001\\0110\end{pmatrix}_{N.} \qquad (8.1)$$

Special cases may be obtained from formula (8.1). Thus, for moments m_{1100}, m_{0011}, we have $a_{1100} = a_{0011} = \frac{1}{n}$, $a_{\substack{1000\\0100}} = a_{\substack{0010\\0001}} = -\frac{1}{n^2}$,

yielding $\mathcal{D}_{1111} = \frac{C_2}{n^2} - \frac{2C_3}{n^3} + \frac{C_4}{n^4}$, $\mathcal{D}_{\substack{1100\\0011}} = \frac{C_{\substack{1\\1}}}{n^2} - \frac{2C_{\substack{2\\1}}}{n^3} + \frac{C_{\substack{2\\2}}}{n^4}$, $\mathcal{D}_{1100} = \mathcal{D}_{0011}$

$= \frac{C_1}{n} - \frac{C_2}{n^2}$, $\mathcal{D}_{\substack{1010\\0101}} = \mathcal{D}_{\substack{1001\\0110}} = \frac{C_{\substack{2\\2}}}{n^4}$. Thus

$$M_{11}(m_{1100}, m_{0011}) = \left(\frac{C_2}{n^2} - \frac{2C_3}{n^3} + \frac{C_4}{n^4} \right) (1111)_N$$

$$+ \left\{ \frac{C_{\substack{1\\1}}}{n^2} - \frac{2C_{\substack{2\\1}}}{n^3} + \frac{C_{\substack{2\\2}}}{n^4} - \left(\frac{C_1}{n} - \frac{C_2}{n^2} \right)^2 \right\} (1100)_N (0011)_N$$

$$+ \left(\frac{C_{\substack{2\\2}}}{n^4} \right) \left\{ (1010)_N (0101)_N + (1001)_N (0110)_N \right\} .$$

When all four variables are identical, this yields the variance of the sample variance

$$M_2(m_2) = \left(\frac{C_2}{n^2} - \frac{2C_3}{n^3} + \frac{C_4}{n^4} \right) (4)_N + \left\{ \frac{C_{\substack{1\\1}}}{n^2} - \frac{2C_{\substack{2\\1}}}{n^3} + \frac{3C_{\substack{2\\2}}}{n^4} - \left(\frac{C_1}{n} - \frac{C_2}{n^2} \right)^2 \right\} (2)_N^2$$

as in [5, p.43] .

With suitable modification, Table 5 can also be used for the calculation of the unbiased estimate of $F_{1100} F_{0011}$. The results can be expressed in terms of sample deviates, so one can use four values of (R), rather than 15 values of (R).

The results are such that the value of the coefficient D_R of any individual term can be computed without regard to other terms - very useful in a field in which the algebra is so complex that occasional errors have been found in the results of even the most accomplished workers.

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