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ON CERTAIN FINITE SERIES INVOLVING THE H-FUNCTION OF TWO VARIABLES

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ABSTRACT

In this paper we obtain two new and interesting finite series for the H-function of two variables in a very elegant form and without having severe restrictions on the parameters involved by a direct and simple method. On account of the most general nature of the H-function of two variables, a number of related finite series for a number of other simple and useful functions can also be obtained as special cases of our results. As an illustration, we obtain here from our main results, the corresponding finite series for Kampé de Fériet function, Apell's function and Gauss' hypergeometric function which are also believed to be new. Thus the present study provides us a number of interesting new results for a large spectrum of special functions in a direct and simple manner.

RESUMEN

En este trabajo damos dos series nuevas para la función H- de dos variables , en una forma elegante y sin muchas restricciones sobre sus parámetros . El método es directo y sencillo . Debido al carácter más general de la función H- de dos variables, los resultados establecidos generan muchos casos especiales con otras funciones . Mencionamos algunos resultados para las funciones de Kampé de Fériet, función de Appell y la función hipergeométrica de Gauss.

1. INTRODUCTION

The parameters of the H-function of two variables[6] occurring in the present paper are displayed in the following contracted notation[5] :

$$H^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(c_j;r_j)_{1,p_2};(e_j,E_j)_{1,p_3}\\(b_j;\beta_j,B_j)_{1,q_1}:(d_j;\delta_j)_{1,q_2};(f_j,F_j)_{1,q_3}\end{matrix}\right]$$

$$= (2\pi i)^{-2}\int_{L_1}\int_{L_2}\phi(s,t)\,\theta_1(s)\,\theta_2(t)\,x^s\,y^t\,ds\,dt \qquad (1.1)$$

where, for convenience, let $(a_j;\alpha_j,A_j)_{n_1+1,p_1}$ and $(c_j,r_j)_{n_2+1,p_2}$ abbreviate the parameters sequence $(a_{n_1+1};\alpha_{n_1+1},A_{n_1+1}),\ldots,(a_{p_1};\alpha_{p_1},A_{p_1})$ and $(c_{n_2+1},r_{n_2+1}),\ldots,(c_{p_2},r_{p_2})$ respectively for the integers n_i and p_i such that $0\leqslant n_i\leqslant p_i$ $(i=1,2)$ and similar interpretations for $(b_j;\beta_j,B_j)_{m_1+1,q_1}$, $(d_j,\delta_j)_{m_2+1,q_2}$ and so on.

Also,

$$\phi(s,t) = \prod_{1}^{n_1}\Gamma(1-a_j+\alpha_j s+A_j t)\left[\prod_{n_1+1}^{p_1}\Gamma(a_j-\alpha_j s-A_j t)\prod_{1}^{q_1}\Gamma(1-b_j+\beta_j s+B_j t)\right]^{-1}$$

$$\theta_1(s) = \prod_{1}^{n_2} \Gamma(1-c_j+r_j s) \prod_{1}^{m_2} \Gamma(d_j-\delta_j s) \left[\prod_{n_2+1}^{p_2} \Gamma(c_j-r_j s) \prod_{m_2+1}^{q_2} \Gamma(1-d_j+\delta_j s) \right]^{-1}$$

and with $\theta_2(t)$ defined analogously in terms of the parameter sets $(e_j, E_j)_{1,p_3}$, $(f_j, F_j)_{1,q_3}$.

The conditions on parameters of the H-function of two variables, its asymptotic expansions, some of its properties, particular cases, nature of contours L_1 and L_2 in (1.1) etc. can be referred to in a paper by Mittal and Gupta[6].

2. SIMPLIFYING NOTATIONS AND MAIN RESULTS

Since, only the parameters subscripted 1 in the definition of the H-function of two variables undergo changes in our main results, therefore, to simplify notational problems, we specify only these parameters in our main finite series. Thus,

$$H\,[\,(a_1-r;\alpha_1 h, \alpha_1 k)\,,\,(b_1-r;\beta_1 h,\beta_1 k)\,]$$

would represent the H-function of two variables defined by (1.1) but having a_1 replaced by a_1-r, α_1 replaced by $\alpha_1 h$, A_1 replaced by $\alpha_1 k$, b_1 replaced by b_1-r, β_1 replaced by $\beta_1 h$, B_1 replaced by $\beta_1 k$ and rest of the parameters being same as in (1.1).

MAIN RESULTS

$$\sum_{r=1}^{n} \beta_1^{r-1} \alpha_1^{n-r} \begin{bmatrix} r-b_1 & 1+r-a_1 \\ \beta_1 & \alpha_1 \end{bmatrix} H\left[(a_1-r;\alpha_1 h,\alpha_1 k)\,,\,(b_1-r;\beta_1 h,\beta_1 k) \right] =$$

$$= \beta_1^n \, H\left[(a_1 - n - 1 ; \alpha_1 h , \alpha_1 k) \;,\; (b_1 - n ; \beta_1 h , \beta_1 k) \right]$$

$$- \alpha_1^n \, H\left[(a_1 - 1 ; \alpha_1 h , \alpha_1 k) \;,\; (b_1 ; \beta_1 h , \beta_1 k) \right] \qquad (2.1)$$

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} H\left[(a_1 - n + r ; \alpha_1 h , \alpha_1 k) \;,\; (b_1 + r ; \beta_1 h , \beta_1 k) \right]$$

$$= (b_1 - a_1 + 1)_n \, H\left[(a_1 ; \alpha_1 h , \alpha_1 k) \;,\; (b_1 ; \beta_1 h , \beta_1 k) \right] \qquad (2.2)$$

PROOFS OF (2.1) AND (2.2)

To prove (2.1), we first give the following simple three term contiguous relation for the H-function of two variables :

$$d(r - b_1 \,,\, 1 - a_1 + r) \, H\left[(a_1 - r ; \alpha_1 h , \alpha_1 k) \;,\; (b_1 - r ; \beta_1 h , \beta_1 k) \right]$$

$$= \alpha_1 \, H\left[(a_1 - r ; \alpha_1 h , \alpha_1 k) \;,\; (b_1 - r + 1 ; \beta_1 h , \beta_1 k) \right]$$

$$- \beta_1 \, H\left[(a_1 - r - 1 ; \alpha_1 h , \alpha_1 k) \;,\; (b_1 - r ; \beta_1 h , \beta_1 k) \right] \qquad (2.3)$$

where $d\,(r - b_1 \,,\, 1 - a_1 + r)$ stands for the determinant

$$\begin{bmatrix} r - b_1 & 1 - a_1 + r \\ \beta_1 & \alpha_1 \end{bmatrix} .$$

To prove (2.3), we note from the definition of the H-function of two variables that the replacement of $(a_1 - r)$ by $(a_1 - r - 1)$ in (1.1) and the application of the recurrence formula $\Gamma(z+1) = z\,\Gamma(z)$ is equivalent to the introduction of the additional multiplying factor $(1 - a_1 + r + \alpha_1 hs + \alpha_1 kt)$ into the contour integral format for the H-function of two variables. Similarly, the replacement of $(b_1 - r)$ by $(b_1 - r + 1)$ will introduce an additional multiplying factor $(-b_1 + r + \beta_1 hs + \beta_1 kt)$. Consequently, we can simply form a following 3-term recurrence relation involving undetermined coefficients A, B and C:

$$A\,H\left[(a_1 - r;\alpha_1 h,\alpha_1 k)\ ,\ (b_1 - r;\beta_1 h,\beta_1 k)\right]$$

$$= B\,H\left[(a_1 - r;\alpha_1 h,\alpha_1 k)\ ,\ (b_1 - r + 1;\beta_1 h,\beta_1 k)\right]$$

$$+ C\,H\left[(a_1 - r - 1;\alpha_1 h,\alpha_1 k)\ ,\ (b_1 - r;\beta_1 h,\beta_1 k)\right] \qquad (2.4)$$

and then require that

$$A = B(-b_1 + r + \beta_1 hs + \beta_1 kt) + C(1 - a_1 + r + \alpha_1 hs + \alpha_1 kt)$$

be an identity in s and t. Hence, A, B and C can be evaluated. On evaluating the values of these quantities, and after a little simplification, we easily arrive at the required result (2.3).

Now, if we multiply both sides of (2.3) by

$$\alpha_1^{n-1}\ ,\ \beta_1\alpha_1^{n-2}\ ,\ \beta_1^2\alpha_1^{n-3}\ ,\ldots,\ \beta_1^{n-2}\alpha_1 \quad\text{and}\quad \beta_1^{n-1}$$

in succession and take the sum, we easily arrive at the required result (2.1).

To prove (2.2) , we consider the following contiguous relation which can easily be proved on lines similar to that of (2.3) :

$$d(a_1 - 1 , b_1 + 1)\ H\left[(a_1 ; \alpha_1 h , \alpha_1 k) \ , \ (b_1 + 1 ; \beta_1 h , \beta_1 k)\right]$$

$$= \alpha_1\ H\left[(a_1 ; \alpha_1 h , \alpha_1 k) \ , \ (b_1 + 2 ; \beta_1 h , \beta_1 k)\right]$$

$$- \beta_1\ H\left[(a_1 - 1 ; \alpha_1 h , \alpha_1 k) \ , \ (b_1 + 1 ; \beta_1 h , \beta_1 k)\right] \qquad (2.5)$$

where $d(a_1 - 1 , b_1 + 1)$ stands for the determinant

$$\begin{bmatrix} a_1 - 1 & b_1 + 1 \\ \alpha_1 & \beta_1 \end{bmatrix} .$$

If we now iterate by expanding each term on the right hand side of (2.5) by the use of this very relation, we get

$$d(a_1 - 1 , b_1)\ d(a_1 - 1 , b_1 + 1)\ d(a_1 - 2 , b_1)\ H\left[(a_1 ; \alpha_1 h , \alpha_1 k) \ , \right.$$
$$\left. (b_1 ; \beta_1 h ; \beta_1 k)\right]$$

$$= \alpha_1^2\ d(a_1 - 2 , b_1)\ H\left[(a_1 ; \alpha_1 h , \alpha_1 k) \ , \ (b_1 + 2 ; \beta_1 h , \beta_1 k)\right]$$

$$- \alpha_1 \beta_1 \left[d(a_1 - 2 , b_1) + d(a_1 - 1 , b_1 + 1)\right] H\left[(a_1 - 1 ; \alpha_1 h , \alpha_1 k) \ , \right.$$

$$\left. (b_1 + 1 ; \beta_1 h , \beta_1 k)\right] + \beta_1^2\ d(a_1 - 1 , b_1 + 1)\ H\left[(a_1 - 2 ; \alpha_1 h , \alpha_1 k) \ , \right.$$
$$\left. (b_1 ; \beta_1 h , \beta_1 k)\right]$$

$$(2.6)$$

Again, by further iterations, we get, expressions of the form

$$\lambda\, H\left[(a_1\,;\,\alpha_1 h,\,\alpha_1 k)\,,\;(b_1\,;\,\beta_1 h,\,\beta_1 k)\right]$$

$$= \sum_{r=0}^{n} \lambda_r\, H\left[(a_1 - n + r\,;\,\alpha_1 h,\,\alpha_1 k)\,,\;(b_1 + r\,;\,\beta_1 h,\,\beta_1 k)\right] \tag{2.7}$$

in which λ's involve sums and products of the determinants.
If we put $\beta_1 = \alpha_1$ in (2.7), the coefficients are greatly simplified and after a little simplification, (2.7) easily reduces to (2.2).

3. SPECIAL CASES

If we specialize the parameters of the various H-functions of two variables involved in the series (2.1) and (2.2), such that all of them reduce to Kampé de Fériet functions[1], we get, after a little simplification and by virtue of a known formula[4], the following interesting results involving Kampé de Fériet functions:

$$\sum_{r=1}^{n} \frac{(a_1-b_1+1)\,\Gamma(a_1+r)}{\Gamma(b_1+r)}\, F^{p_1,p_2}_{q_1,q_2}\left[\begin{matrix} a_1+r,\,(a_j)_{2,p_1} : (c_j)_{1,p_2}\,;\,(e_j)_{1,p_2} \\ b_1+r,\,(b_j)_{2,q_1} : (d_j)_{1,q_2}\,;\,(f_j)_{1,q_2}\end{matrix}\,\middle|\, x,y\right]$$

$$= \frac{\Gamma(a_1+n+1)}{\Gamma(b_1+n)}\, F^{p_1,p_2}_{q_1,q_2}\left[\begin{matrix} a_1+n+1,\,(a_j)_{2,p_1} : (c_j)_{1,p_2}\,;\,(e_j)_{1,p_2} \\ b_1+n\;\;,\,(b_j)_{2,q_1} : (d_j)_{1,q_2}\,;\,(f_j)_{1,q_2}\end{matrix}\,\middle|\, x,y\right]$$

$$- \frac{\Gamma(a_1+1)}{\Gamma(b_1)}\, F^{p_1,p_2}_{q_1,q_2}\left[\begin{matrix} a_1+1,\,(a_j)_{2,p_1} : (c_j)_{1,p_2}\,;\,(e_j)_{1,p_2} \\ (b_j)_{1,q_1} : (d_j)_{1,q_2}\,;\,(f_j)_{1,q_2}\end{matrix}\,\middle|\, x,y\right] \tag{3.1}$$

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{\Gamma(a_1+n-r)}{\Gamma(b_1-r)}$$

$$F^{p_1,p_2}_{q_1,q_2}\left| \begin{matrix} a_1+n-r, & (a_j)_{2,p_1} & : (c_j)_{1,p_2} ; (e_j)_{1,p_2} \\ b_1-r & , (b_j)_{2,q_1} & : (d_j)_{1,q_2} ; (f_j)_{1,q_2} \end{matrix} \right| x,y \Big|$$

$$= \frac{\Gamma(a_1)}{\Gamma(b_1)} (a_1-b_1+1)_n \, F^{p_1,p_2}_{q_1,q_2}\left| \begin{matrix} (a_j)_{1,p_1} : (c_j)_{1,p_2} ; (e_j)_{1,p_2} \\ (b_j)_{1,q_1} : (d_j)_{1,q_2} ; (f_j)_{1,q_2} \end{matrix} \right| x,y \Big| \tag{3.2}$$

Also, if we put $p_1 = q_1 = p_2 = 1$ and $q_2 = 0$ in the equations (3.1) and (3.2), we get, after a little simplification, the following series involving Appell's functions respectively:

$$(1+a-b) \sum_{r=1}^{n} \frac{\Gamma(a+r)}{\Gamma(b+r)} F_1(a+r ; \alpha,\beta ; b+r ; x,y)$$

$$= \frac{\Gamma(a+n+1)}{\Gamma(b+n)} F_1(a+n+1 ; \alpha,\beta ; b+n ; x,y)$$

$$- \frac{\Gamma(a+1)}{\Gamma(b)} F_1(a+1 ; \alpha,\beta ; b ; x,y) \tag{3.3}$$

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{\Gamma(a+n-r)}{\Gamma(b-r)} F_1(a+n-r ; \alpha,\beta ; b-r ; x,y)$$

$$= (a-b+1)_n \frac{\Gamma(a)}{\Gamma(b)} F_1(a ; \alpha,\beta ; b ; x,y) \tag{3.4}$$

Again, if in the equations (3.3) and (3.4), we put $y = x$, and use a known result[2], we get, after a little simplification, the following interesting series involving Gauss' hypergeometric functions[3] respectively:

$$(1+a-b) \sum_{r=1}^{n} \frac{\Gamma(a+r)}{\Gamma(b+r)} \; {}_2F_1\left(\begin{matrix} a+r, \alpha \\ b+r \end{matrix}; x\right)$$

$$= \frac{\Gamma(a+n+1)}{\Gamma(b+n)} \; {}_2F_1\left(\begin{matrix} a+n+1, \alpha \\ b+n \end{matrix}; x\right) - \frac{\Gamma(1+a)}{\Gamma(b)} \; {}_2F_1\left(\begin{matrix} 1+a, \alpha \\ b \end{matrix}; x\right) \tag{3.5}$$

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{\Gamma(a+n-r)}{\Gamma(b-r)} \; {}_2F_1\left(\begin{matrix} a+n-r, \alpha \\ b-r \end{matrix}; x\right)$$

$$= (a-b+1)_n \frac{\Gamma(a)}{\Gamma(b)} \; {}_2F_1\left(\begin{matrix} a, \alpha \\ b \end{matrix}; x\right) \tag{3.6}$$

Related finite series for other special functions can also be obtained from (2.1) and (2.2) by reducing the H-function of two variables into some other simpler functions.

REFERENCES

[1] APPELL, P. and KAMPE DE FERIET, J.: *"Functions Hypergeometriques et Hyperspheriques"*, Gauthier-Villars, París (1926).

[2] BAILEY, W.N.: *"Generalized Hypergeometric Series"*, Cambridge Univ. Press (1935).

[3] ERDELYI, A. et al.: *"Higher Transcendental Functions"*, Vol. 1, Mc Graw-Hill Book. Co., Inc. New York (1953).

[4] GARG, O.P.: *"On Recurrence Relations for the H-function of Two Variables"*, Proc. Indian Acad. Sci., 84 A (1976), 244-249.

[5] GUPTA, K.C. and GOYAL, S.P.: *"Unified Summation Formula for the H-function of Two Variables"*, Jnanabha, 7 (1977) (in press).

[6] MITTAL, P.K. and GUPTA, K.C.: *"An Integral Involving Generalized Function of Two Variables"*, Proc. Indian Acad. Sci., 75 A (1972), 117-123.