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An application of an inequality of J. M. Aldaz

Una aplicación de una desigualdad de J. M. Aldaz

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Abstract

The aim of this paper is a to give a new proof that Hölder inequality is implied by the Cauchy-Schwarz inequality. Our proof is short and is based on the use of an inequality obtained by J. M. Aldaz in the paper: A stability version of Hölder's inequality, Journal of Mathematical Analysis and Applications, 343, 2(2008), 842–852.

Key words and phrases: Inequalities, Young's inequality, Cauchy-Schwarz inequality, Hölder's inequality.

Resumen

La finalidad de este artículo es dar una nueva demostración de que la desigualdad de Cauchy-Schwarz implica las desigualdades de Hölder. Para establecer nuestro resultado, utilizamos una desigualdad obtenida por J. M. Aldaz en su art??culo: A stability version of Hölder's inequality, Journal of Mathematical Analysis and Applications, 343, 2 (2008), 842–852.

Palabras y frases clave: Desigualdades, desigualdad de Young, desigualdad de Cauchy-Schwarz, desigualdad de Hölder.

1 Introduction

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space (μ is a positive measure). For all mesurable functions $f, g : \Omega \mapsto \mathbb{C}$ on Ω, we recall the Hölder's inequality:

$$\int_\Omega |fg| d\mu \leq \Big(\int_\Omega |f|^p d\mu \Big)^{\frac{1}{p}} \Big(\int_\Omega |f|^q d\mu \Big)^{\frac{1}{q}}, \ \forall p, q \geq 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1. \tag{H}$$

If $p = q = 2$ then we obtain the Cauchy-Schwarz inequality:

$$\int_\Omega |fg| d\mu \leq \Big(\int_\Omega |f|^2 d\mu \Big)^{\frac{1}{2}} \Big(\int_\Omega |f|^2 d\mu \Big)^{\frac{1}{2}}. \tag{C.S}$$

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Their discrete versions are respectively, given by:

$$\sum_{i=1}^{n}|x_iy_i| \leq \left[\sum_{i=1}^{n}|x_i|^p\right]^{\frac{1}{p}}\left[\sum_{i=1}^{n}|y_i|^q\right]^{\frac{1}{q}} := \|x\|_p\,\|y\|_q\,, \qquad (H)_d$$

and

$$\sum_{i=1}^{n}|x_iy_i| \leq \left[\sum_{i=1}^{n}|x_i|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}|y_i|^2\right]^{\frac{1}{2}} := \|x\|_2\,\|y\|_2\,, \qquad (C.S)_d$$

for all positive integer n and all vectors $(x_1,\ldots,x_n),(y_1,\ldots,y_n) \in \mathbb{K}^n$, where the field $\mathbb{K}$ is real or complex.

Easily, we have $(H) \Longrightarrow (C.S)$.

It is natural to raise the question: is the converse true?.

Many connections between classical discrete inequalities were investigated in the book [8], where in particular the equivalence $(H)_d \Longleftrightarrow (C.S)_d$ was deducted through several intermidiate results.

Also, we notice that A. W. Marshall and I. Olkin pointed out in their book [7] that the Cauchy-Schwarz inequality implies Lyapunov's inequality which itself implies the arithmetic-geometric mean inequality. Their discussions led to the conclusions that, in a sense, the arithmetic-geometric mean inequality, Hölder's inequality, the Cauchy-Schwarz inequality, and Lyapunov's inequality are all equivalent [7, p. 457].

In 2006, Y-C Li and S-Y Shaw [6] gave a proof of Hölder's inequality by using the Cauchy-Schwarz inequality. Their method lies on the fact that the convexity of a function on an open and finite interval is equivalent to continuity and midconvexity.

In 2007, the equivalence between the integral inequalities (H) and $(C-S)$ was studied by C. Finol and M. Wójtowicz in [4]. They gave a proof that $(C-S)$ implies (H) by using density arguments, induction and the conclusions were obtained after three steps of proof.

For many other results concerning to the implication $(C-S) \Longrightarrow (H)$ in the discrete case, the reader is invited to see for instance [4, 5, 6, 7, 8].

Recently (see [1]), the author gave a proof of the implication $(C-S) \Longrightarrow (H)$ by using an improvement of Young's inequality.

The aim of this paper is to provide a new (and short) proof of the implication $(H) \Longrightarrow (C.S)$. Our method is quite different from those used in [6] and [4]. Our method is based on the following result of J. M. Aldaz (see [2]).

Theorem 1.1. *Let $1 < p < \infty$ and let $q = \dfrac{p}{p-1}$ be its conjugate exponent. If $f \in L^p$, $g \in L^q$, $\|f\|_p, \|g\|_q > 0$, and $1 < p \leq 2$, then*

$$\|f\|_p\|g\|_q\left(1-\frac{1}{p}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{q/2}}{\|g\|_q^{q/2}}\right\|_2^2\right)_+ \leq \|fg\|_1 \leq \|f\|_p\|g\|_q\left(1-\frac{1}{q}\left\|\frac{|f|^{p/2}}{\|f\|_p^{p/2}}-\frac{|g|^{q/2}}{\|g\|_q^{q/2}}\right\|_2^2\right),$$

while if $2 \leq p < \infty$, the terms $\frac{1}{p}$ and $\frac{1}{q}$ exchange their positions in the preceding inequalities.

In Theorem 1.1, $t_+ = \max\{t, 0\}$ for any real number t. As a consequence of Theorem 1.1, we conclude the following inequality:

$$\int_\Omega |fg|\,d\mu \;\leq\; ||f||_p||g||_q\left(1-\frac{1}{\max\{p,q\}}\left\|\frac{|f|^{\frac{p}{2}}}{||f||_p^{\frac{p}{2}}}-\frac{|g|^{\frac{q}{2}}}{||g||_q^{\frac{q}{2}}}\right\|_2^2\right), \qquad (1)$$

for all $f \in L^p$, $g \in L^q$, $\|f\|_p, \|g\|_q > 0$, and for all $1 < p < \infty$ with $q = \dfrac{p}{p-1}$ is its conjugate exponent.

2 Proof of the implication: $(C-S) \Longrightarrow (H)$

We avoid the trivial cases, so we suppose that $1 < p, q$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. We suppose also that $||f||_p \neq 0$ and $||g||_q \neq 0$.

We set $u = \frac{|f|^{\frac{p}{2}}}{||f||_p^{\frac{p}{2}}}$ and $v = \frac{|g|^{\frac{q}{2}}}{||g||_q^{\frac{q}{2}}}$, then u and v are unit vectors in the real Hilbert space $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$. We recall that the inner product of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$ is given by

$$< f \mid g > := \int_\Omega f(x) g(x) d\mu(x),$$

for all $f, g \in L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$.

According to the inequality (1) and the usual Cauchy-Schwarz inequality in the real Hilbert space $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$, we have successively,

$$\begin{aligned} \int_\Omega |f(x)g(x)| d\mu(x) &\leq ||f||_p ||g||_q \left(1 - \frac{1}{\max\{p,q\}} \|u - v\|^2\right) \\ &= ||f||_p ||g||_q \left(1 - \frac{1}{\max\{p,q\}} (\|u\|^2 + \|v\|^2 - 2 < u \mid v >)\right) \\ &= ||f||_p ||g||_q \left(1 - \frac{2}{\max\{p,q\}} + \frac{2}{\max\{p,q\}} < u \mid v >\right) \\ &\leq ||f||_p ||g||_q \left(1 - \frac{2}{\max\{p,q\}} + \frac{2}{\max\{p,q\}}\right) = ||f||_p ||g||_q. \end{aligned} \tag{2}$$

This end the proof.

Remark 2.1. 1. The inequality (2) shows that the equality in Holder's inequality holds if and only if

$$\frac{|f|^{\frac{p}{2}}}{||f||_p^{\frac{p}{2}}} = \frac{|g|^{\frac{q}{2}}}{||g||_q^{\frac{q}{2}}} \quad \mu - \text{a.e.}$$

That is $|f|^p \, ||g||_q^q = |g|^q \, ||f||_p^p$, μ-a.e. on Ω.

2. In [1], for all $f \in L^p \setminus \{0\}$ and all $g \in L^q \setminus \{0\}$, the following inequality was obtained by using certain improvements to Young's inequality:

$$\int_\Omega |fg| d\mu \leq \Big(\frac{1}{p^2} + \frac{1}{q^2}\Big) ||f||_p ||g||_q + \frac{2}{pq} ||f||_p^{1-\frac{p}{2}} ||g||_q^{1-\frac{q}{2}} \int_\Omega |f|^{p/2} |g|^{q/2} d\mu. \tag{3}$$

It is easy to see that the inequality (3) is equivalent to the following inequality:

$$\int_\Omega |fg| \, d\mu \leq ||f||_p ||g||_q \left(1 - \frac{1}{pq} \left\| \frac{|f|^{\frac{p}{2}}}{||f||_p^{\frac{p}{2}}} - \frac{|g|^{\frac{q}{2}}}{||g||_q^{\frac{q}{2}}} \right\|_2^2 \right), \tag{4}$$

for all $f \in L^p \setminus \{0\}$ and all $g \in L^q \setminus \{0\}$.

The inequality (4) is a variant of the inequality (1). It was obtained by J. M. Aldaz [3] in a different manner.

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