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Insertion of a contra-Baire-1 (Baire-.5) function

Inserción de una función Contra-Baire-1 (Baire-.5)

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Abstract

A necessary and sufficient condition in terms of lower cut sets are given for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that F_σ-kernel of sets are F_σ-sets.

Key words and phrases: Insertion, strong binary relation, Baire-.5 function, kernel of sets, lower cut set.

Resumen

Se proporciona una condición necesaria y suficiente en términos de conjuntos de cortes inferiores para la inserción de una función Baire-.5 entre dos funciones comparables de valores reales en los espacios topológicos donde el F_σ-kernel de los conjuntos es F_σ-sets.

Palabras y frases clave: Inserción, relación binaria fuerte, funcin Baire-.5, núcleo de conjuntos, conjunto de corte inferior.

1 Introduction

A generalized class of closed sets was considered by Maki in 1986 [16]. He investigated the sets that can be represented as union of closed sets and called them V-sets. Complements of V-sets, i.e., sets that are intersection of open sets are called Λ-sets [16].

Recall that a real-valued function f defined on a topological space X is called A-continuous [21] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of A-continuity. However, for unknown concepts the reader may refer to [4, 10]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

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J. Dontchev in [5] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 7, 8, 9, 11, 12, 20].

Results of Katětov [13, 14] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient condition for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that F_σ-kernel of sets are F_σ-sets.

A real-valued function f defined on a topological space X is called *contra-Baire-1 (Baire-.5)* if the preimage of every open subset of $\mathbb{R}$ is a G_δ-set in X [22]. If g and f are real-valued functions defined on a space X, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all x in X.

The following definitions are modifications of conditions considered in [15].

A property P defined relative to a real-valued function on a topological space is a *$B-.5$-property* provided that any constant function has property P and provided that the sum of a function with property P and any Baire-.5 function also has property P. If P_1 and P_2 are $B-.5$-properties, the following terminology is used:

(i) A space X has the *weak $B-.5$-insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a Baire-.5 function h such that $g \leq h \leq f$.

(ii) A space X has the *$B-.5$-insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$ has property P_1 and f has property P_2, then there exists a Baire-.5 function h such that $g < h < f$.

In this paper, for a topological space that F_σ-kernel of sets are F_σ-sets, is given a sufficient condition for the weak $B-.5$-insertion property. Also for a space with the weak $B-.5$-insertion property, we give a necessary and sufficient condition for the space to have the $B-.5$-insertion property. Several insertion theorems are obtained as corollaries of these results.

2 The Main Result

Before giving a sufficient condition for insertability of a Baire-.5 function, the necessary definitions and terminology are stated.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V as follows:

$$A^\Lambda = \bigcap\{O : O \supseteq A,\ O \in (X,\tau)\} \quad \text{and} \quad A^V = \bigcup\{F : F \subseteq A,\ F^c \in (X,\tau)\}.$$

In [6, 17, 19], A^Λ is called the *kernel* of A.

We define the subsets $G_\delta(A)$ and $F_\sigma(A)$ as follows:

$$G_\delta(A) = \bigcup\{O : O \subseteq A,\ O \text{ is } G_\delta\text{-set}\} \quad \text{and} \quad F_\sigma(A) = \bigcap\{F : F \supseteq A, F \text{ is } F_\sigma\text{-set}\}$$

$F_\sigma(A)$ is called the *F_σ-kernel* of A.

The following Lemma is a direct consequence of the definition F_σ-kernel of sets.

Lemma 2.1. *The following conditions on the space X are equivalent:*

(i) For every G of G_δ-set we have $F_\sigma(G)$ is a G_δ-set.

(ii) For each pair of disjoint G_δ-sets as G_1 and G_2 we have $F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset$.

The following first two definitions are modifications of conditions considered in [13, 14].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \ \bar{\rho} \ y$ if and only if $y \ \rho \ \nu$ implies $x \ \rho \ \nu$ and $u \ \rho \ x$ implies $u \ \rho \ y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1. If $A_i \ \rho \ B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i \ \rho \ C$ and $C \ \rho \ B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2. If $A \subseteq B$, then $A \ \bar{\rho} \ B$.

3. If $A \ \rho \ B$, then $F_\sigma(A) \subseteq B$ and $A \subseteq G_\delta(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main results:

Theorem 2.1. *Let g and f be real-valued functions on the topological space X, that F_σ-kernel of sets in X are $F_\sigma-$ sets , with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \ \rho \ A(g, t_2)$, then there exists a Baire-.5 function h defined on X such that $g \leq h \leq f$.*

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \ \rho \ A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1) \ \bar{\rho} \ F(t_2), G(t_1) \ \bar{\rho} \ G(t_2)$, and $F(t_1) \ \rho \ G(t_2)$. By Lemmas 1 and 2 of [14] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1) \ \rho \ H(t_2), H(t_1) \ \rho \ H(t_2)$ and $H(t_1) \ \rho \ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$. We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have

$$h^{-1}(t_1, t_2) = G_\delta(H(t_2)) \setminus F_\sigma(H(t_1)).$$

Hence $h^{-1}(t_1, t_2)$ is a G_δ-set in X, i.e., h is a Baire-.5 function on X. □

The above proof used the technique of Theorem 1 of [13].

Theorem 2.2. *Let P_1 and P_2 be $B-.5$-property and X be a space that satisfies the weak $B-.5$-insertion property for (P_1, P_2). Also assume that g and f are functions on X such that $g < f, g$ has property P_1 and f has property P_2. The space X has the $B-.5$-insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f-g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each $n, X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are completely separated by Baire$-.5$ functions.*

Proof. Theorem 2.1 of [18]. □

3 Applications

Definition 3.1. A real-valued function f defined on a space X is called *contra-upper semi-Baire-.5* (resp. *contra-lower semi-Baire-.5*) if $f^{-1}(-\infty, t)$ (resp. $f^{-1}(t, +\infty)$) is a $G_\delta-$set for any real number t.

The abbreviations $usc, lsc, cusB-.5$ and $clsB-.5$ are used for upper semicontinuous, lower semicontinuous, contra-upper semi-Baire-.5, and contra-lower semi-Baire-.5, respectively.

Remark 3.1. [13, 14]. A space X has the weak c-insertion property for (usc, lsc) if and only if X is normal.

Before stating the consequences of Theorems 2.1 and 2.2 we suppose that X is a topological space that F_σ-kernel of sets are F_σ-sets.

Corollary 3.1. *For each pair of disjoint F_σ-sets F_1, F_2, there are two G_δ-sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ if and only if X has the weak $B-.5$-insertion property for $(cusB-.5, clsB-.5)$.*

Proof. Let g and f be real-valued functions defined on the X, such that f is lsB_1, g is usB_1, and $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $F_\sigma(A) \subseteq G_\delta(B)$, then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a F_σ-set and since $\{x \in X : g(x) < t_2\}$ is a G_δ-set, it follows that $F_\sigma(A(f, t_1)) \subseteq G_\delta(A(g, t_2))$. Hence $t_1 < t_2$ implies that $A(f, t_1)\ \rho\ A(g, t_2)$. The proof follows from Theorem 2.1.

On the other hand, let F_1 and F_2 are disjoint F_σ-sets. Set $f = \chi_{F_1^c}$ and $g = \chi_{F_2}$, then f is $clsB-.5$, g is $cusB-.5$, and $g \leq f$. Thus there exists Baire-.5 function h such that $g \leq h \leq f$. Set $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$, then G_1 and G_2 are disjoint G_δ-sets such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$. □

Remark 3.2. [23]. A space X has the weak c-insertion property for (lsc, usc) if and only if X is extremally disconnected.

Corollary 3.2. *For every G of G_δ-set, $F_\sigma(G)$ is a G_δ-set if and only if X has the weak $B-.5$-insertion property for $(clsB-.5, cusB-.5)$.*

Proof. Let g and f be real-valued functions defined on the X, such that f is $clsB-.5$, g is $cusB-.5$, and $f \leq g$. If a binary relation ρ is defined by $A\ \rho\ B$ in case $F_\sigma(A) \subseteq G \subseteq F_\sigma(G) \subseteq G_\delta(B)$ for some G_δ-set g in X, then by hypothesis and Lemma 2.1 ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(g,t_1) = \{x \in X : g(x) < t_1\} \subseteq \{x \in X : f(x) \leq t_2\} = A(f,t_2);$$

since $\{x \in X : g(x) < t_1\}$ is a G_δ-set and since $\{x \in X : f(x) \leq t_2\}$ is a F_σ-set, by hypothesis it follows that $A(g,t_1)\ \rho\ A(f,t_2)$. The proof follows from Theorem 2.1.

On the other hand, Let G_1 and G_2 are disjoint G_δ-sets. Set $f = \chi_{G_2}$ and $g = \chi_{G_1^c}$, then f is $clsB-.5$, g is $cusB-.5$, and $f \leq g$.

Thus there exists Baire-.5 function h such that $f \leq h \leq g$. Set $F_1 = \{x \in X : h(x) \leq \frac{1}{3}\}$ and $F_2 = \{x \in X : h(x) \geq 2/3\}$ then F_1 and F_2 are disjoint F_σ-sets such that $G_1 \subseteq F_1$ and $G_2 \subseteq F_2$. Hence $F_\sigma(F_1) \cap F_\sigma(F_2) = \emptyset$. □

Before starting the consequences of Theorem 2.2, we state and prove some necessary lemmas.

Lemma 3.1. *The following conditions on the space X are equivalent:*

(i) Every two disjoint F_σ-sets of X can be separated by G_δ-sets of X.

(ii) If F is a F_σ-set of X which is contained in a G_δ-set G, then there exists a G_δ-set H such that $F \subseteq H \subseteq F_\sigma(H) \subseteq G$.

Proof. *(i) ⇒ (ii).* Suppose that $F \subseteq G$, where F and G are F_σ-set and G_δ-set of X, respectively. Hence, G^c is a F_σ-set and $F \cap G^c = \emptyset$.

By *(i)* there exists two disjoint G_δ-sets G_1, G_2 such that $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \emptyset \Rightarrow G_1 \subseteq G_2^c$$

hence

$$F \subseteq G_1 \subseteq G_2^c \subseteq G$$

and since G_2^c is a F_σ-set containing G_1 we conclude that $F_\sigma(G_1) \subseteq G_2^c$, i.e.,

$$F \subseteq G_1 \subseteq F_\sigma(G_1) \subseteq G.$$

By setting $H = G_1$, condition *(ii)* holds.

(ii) ⇒ (i). Suppose that F_1, F_2 are two disjoint F_σ-sets of X.

This implies that $F_1 \subseteq F_2^c$ and F_2^c is a G_δ-set. Hence by *(ii)* there exists a G_δ-set H such that, $F_1 \subseteq H \subseteq F_\sigma(H) \subseteq F_2^c$. But

$$H \subseteq F_\sigma(H) \Rightarrow H \cap (F_\sigma(H))^c = \emptyset$$

and

$$F_\sigma(H) \subseteq F_2^c \Rightarrow F_2 \subseteq (F_\sigma(H))^c.$$

Furthermore, $(F_\sigma(H))^c$ is a G_δ-set of X. Hence $F_1 \subseteq H, F_2 \subseteq (F_\sigma(H))^c$ and $H \cap (F_\sigma(H))^c = \emptyset$. This means that condition *(i)* holds. □

Lemma 3.2. *Suppose that X is the topological space such that we can separate every two disjoint $F_\sigma-$sets by $G_\delta-$sets. If F_1 and F_2 are two disjoint $F_\sigma-$sets of X, then there exists a Baire-.5 function $h : X \to [0,1]$ such that $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.*

Proof. Suppose F_1 and F_2 are two disjoint F_σ-sets of X. Since $F_1 \cap F_2 = \emptyset$, hence $F_1 \subseteq F_2^c$. In particular, since F_2^c is a G_δ-set of X containing F_1, by Lemma 3.1, there exists a G_δ-set $H_{1/2}$ such that,

$$F_1 \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq F_2^c.$$

Note that $H_{1/2}$ is a G_δ-set and contains F_1, and F_2^c is a G_δ-set and contains $F_\sigma(H_{1/2})$. Hence, by Lemma 3.1, there exists G_δ-sets $H_{1/4}$ and $H_{3/4}$ such that,

$$F_1 \subseteq H_{1/4} \subseteq F_\sigma(H_{1/4}) \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq H_{3/4} \subseteq F_\sigma(H_{3/4}) \subseteq F_2^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators aremexponents of 2, we obtain G_δ-sets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin F_2$ and $h(x) = 1$ for $x \in F_2$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., h maps X into $[0,1]$. Also, we note that for any $t \in D, F_1 \subseteq H_t$; hence $h(F_1) = \{0\}$. Furthermore, by definition, $h(F_2) = \{1\}$. It remains only to prove that h is a Baire-.5 function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \emptyset$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}$. Hence, they are G_δ-sets of X. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$ then $\{x \in X : h(x) > \alpha\} = \cup\{(F_\sigma(H_t))^c : t > \alpha\}$ hence, every of them is a G_δ-set. Consequently h is a Baire-.5 function. □

Lemma 3.3. *Suppose that X is the topological space such that every two disjoint $F_\sigma-$sets can be separated by $G_\delta-$sets. The following conditions are equivalent:*

(i) Every countable convering of G_δ-sets of X has a refinement consisting of G_δ-sets such that, for every $x \in X$, there exists a G_δ-set containing x such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence $\{F_n\}$ of F_σ-sets with empty intersection there exists a decreasing sequence $\{G_n\}$ of G_δ-sets such that, $\bigcap_{n=1}^\infty G_n = \emptyset$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$.

Proof. *(i) ⇒ (ii).* suppose that $\{F_n\}$ be a decreasing sequence of F_σ-sets with empty intersection. Then $\{F_n^c : n \in \mathbb{N}\}$ is a countable covering of G_δ-sets. By hypothesis *(i)* and Lemma **??**, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every V_n is a G_δ-set and $F_\sigma(V_n) \subseteq F_n^c$. By setting $F_n = (F_\sigma(V_n))^c$, we obtain a decreasing sequence of G_δ-sets with the required properties.

(ii) ⇒ (i). Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of G_δ-sets, we set for $n \in \mathbb{N}$, $F_n = (\bigcup_{i=1}^n H_i)^c$. Then $\{F_n\}$ is a decreasing sequence of F_σ-sets with empty intersection. By *(ii)* there exists a decreasing sequence $\{G_n\}$ consisting of G_δ-sets such that, $\bigcap_{n=1}^\infty G_n = \emptyset$ and for every $n \in \mathbb{N}$, $F_n \subseteq G_n$. Now we define the subsets W_n of X in the following manner:

- W_1 is a G_δ-set of X such that $G_1^c \subseteq W_1$ and $F_\sigma(W_1) \cap F_1 = \emptyset$.
- W_2 is a $G_\delta-$set of X such that $F_\sigma(W_1) \cup G_2^c \subseteq W_2$ and $F_\sigma(W_2) \cap F_2 = \emptyset$, and so on. (By Lemma 3.1, W_n exists).

Then since $\{G_n^c : n \in \mathbb{N}\}$ is a covering for X, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for X consisting of G_δ-sets. Moreover, we have

1. $F_\sigma(W_n) \subseteq W_{n+1}$.
2. $G_n^c \subseteq W_n$.
3. $W_n \subseteq \bigcup_{i=1}^n H_i$.

Now suppose that $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus F_\sigma(W_{n-1})$. Then since $F_\sigma(W_{n-1}) \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of G_δ-sets and covers X. Furthermore, $S_i \cap S_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Finally, consider the following sets:

$$S_1 \cap H_1, \quad S_1 \cap H_2$$
$$S_2 \cap H_1, \quad S_2 \cap H_2, \quad S_2 \cap H_3$$
$$S_3 \cap H_1, \quad S_3 \cap H_2, \quad S_3 \cap H_3, \quad S_3 \cap H_4$$

and continue ad infinitum. These sets are G_δ-sets, cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a G_δ-set containing x that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i+1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are G_δ-sets, and for every point in X we can find a G_δ-set containing the point that intersects only finitely many elements of that refinement. □

Remark 3.3. [13, 14]. A space X has the c-insertion property for (usc, lsc) if and only if X is normal and countably paracompact.

Corollary 3.3. *X has the $B-.5$-insertion property for $(cusB-.5, clsB-.5)$ if and only if every two disjoint F_σ-sets of X can be separated by G_δ-sets, and in addition, every countable covering of G_δ-sets has a refinement that consists of G_δ-sets such that, for every point of X we can find a G_δ-set containing that point such that, it intersects only a finite number of refining members.*

Proof. Suppose that F_1 and F_2 are disjoint F_σ-sets. Since $F_1 \cap F_2 = \emptyset$, it follows that $F_2 \subseteq F_1^c$. We set $f(x) = 2$ for $x \in F_1^c$, $f(x) = \frac{1}{2}$ for $x \notin F_1^c$, and $g = \chi_{F_2}$. Since F_2 is a F_σ-set, and F_1^c is a G_δ-set, therefore g is $cusB-.5$, f is $clsB-.5$ and furthermore $g < f$. Hence by hypothesis there exists a Baire-.5 function h such that, $g < h < f$. Now by setting $G_1 = \{x \in X : h(x) < 1\}$ and $G_2 = \{x \in X : h(x) > 1\}$. We can say that G_1 and G_2 are disjoint G_δ-sets that contain F_1 and F_2, respectively. Now suppose that $\{F_n\}$ is a decreasing sequence of F_σ-sets with empty intersection. Set $F_0 = X$ and define for every $x \in F_n \setminus F_{n+1}$, $f(x) = \frac{1}{n+1}$. Since $\bigcap_{n=0}^\infty F_n = \emptyset$ and for every $x \in X$, there exists $n \in \mathbb{N}$, such that, $x \in F_n \setminus F_{n+1}$, f is well defined. Furthermore, for every $r \in \mathbb{R}$, if $r \leq 0$ then $\{x \in X : f(x) > r\} = X$ is a G_δ-set and if $r > 0$ then by Archimedean property of $\mathbb{R}$, we can find $i \in \mathbb{N}$ such that $\frac{1}{i+1} \leq r$. Now suppose that k is the least natural number such that $\frac{1}{k+1} \leq r$. Hence $\frac{1}{k} > r$ and consequently, $\{x \in X : f(x) > r\} = X \setminus F_k$ is a G_δ-set. Therefore, f is $clsB-.5$. By setting $g = 0$, we have g is $cusB-.5$ and $g < f$. Hence by hypothesis there exists a Baire-.5 function h on X such that, $g < h < f$.

By setting $G_n = \{x \in X : h(x) < \frac{1}{n+1}\}$, we have G_n is a G_δ-set. But for every $x \in F_n$, we have $f(x) \leq \frac{1}{n+1}$ and since $g < h < f$ therefore $0 < h(x) < \frac{1}{n+1}$, i.e., $x \in G_n$ therefore $F_n \subseteq G_n$ and since $h > 0$ it follows that $\bigcap_{n=1}^\infty G_n = \emptyset$. Hence by Lemma 3.3, the conditions holds.

On the other hand, since every two disjoint F_σ-sets can be separated by G_δ-sets, therefore by Corollary 3.1, X has the weak $B-.5$-insertion property for $(cusB-.5, clsB-.5)$. Now suppose that f and g are real-valued functions on X with $g<f$, such that, g is $cusB-.5$ and f is $clsB-.5$. For every $n\in\mathbb{N}$, set

$$A(f-g,3^{-n+1})=\{x\in X:(f-g)(x)\leq 3^{-n+1}\}.$$

Since g is $cusB-.5$, and f is $clsB-.5$, therefore $f-g$ is $clsB-.5$. Hence $A(f-g,3^{-n+1})$ is a F_σ-set of X. Consequently, $\{A(f-g,3^{-n+1})\}$ is a decreasing sequence of F_σ-sets and furthermore since $0<f-g$, it follows that $\bigcap_{n=1}^{\infty}A(f-g,3^{-n+1})=\emptyset$. Now by Lemma 3.3, there exists a decreasing sequence $\{D_n\}$ of G_δ-sets such that $A(f-g,3^{-n+1})\subseteq D_n$ and $\bigcap_{n=1}^{\infty}D_n=\emptyset$. But by Lemma 3.2, $A(f-g,3^{-n+1})$ and $X\setminus D_n$ of F_σ-sets can be completely separated by Baire-.5 functions. Hence by Theorem 2.2, there exists a Baire-.5 function h defined on X such that, $g<h<f$, i.e., X has the $B-.5-$insertion property for $(cusB-.5, clsB-.5)$. □

Remark 3.4. [15]. A space X has the c-insertion property for (lsc, usc) iff X is extremally disconnected and if for any decreasing sequence $\{G_n\}$ of open subsets of X with empty intersection there exists a decreasing sequence $\{F_n\}$ of closed subsets of X with empty intersection such that $G_n\subseteq F_n$ for each n.

Corollary 3.4. *For every G of G_δ-set, $F_\sigma(G)$ is a G_δ-set and in addition for every decreasing sequence $\{G_n\}$ of G_δ-sets with empty intersection, there exists a decreasing sequence $\{F_n\}$ of F_σ-sets with empty intersection such that for every $n\in\mathbb{N}, G_n\subseteq F_n$ if and only if X has the $B-.5$-insertion property for $(clsB-.5, cusB-.5)$.*

Proof. Since for every G of G_δ-set, $F_\sigma(G)$ is a G_δ-set, therefore by Corollary 3.2, X has the weak $B-.5$-insertion property for $(clsB-.5, cusB-.5)$. Now suppose that f and g are real-valued functions defined on X with $g<f, g$ is $clsB-.5$, and f is $cusB-.5$. Set $A(f-g,3^{-n+1})=\{x\in X:(f-g)(x)<3^{-n+1}\}$. Then since $f-g$ is $cusB-.5$, hence $\{A(f-g,3^{-n+1})\}$ is a decreasing sequence of G_δ-sets with empty intersection. By hypothesis, there exists a decreasing sequence $\{D_n\}$ of F_σ-sets with empty intersection such that, for every $n\in\mathbb{N}$, $A(f-g,3^{-n+1})\subseteq D_n$. Hence $X\setminus D_n$ and $A(f-g,3^{-n+1})$ are two disjoint G_δ-sets and therefore by Lemma 2.1, we have

$$F_\sigma(A(f-g,3^{-n+1}))\cap F_\sigma((X\setminus D_n))=\emptyset$$

and therefore by Lemma 3.2, $X\setminus D_n$ and $A(f-g,3^{-n+1})$ are completely separable by Baire-.5 functions. Therefore by Theorem 2.2, there exists a Baire-.5 function h on X such that, $g<h<f$, i.e., X has the $B-.5$-insertion property for $(clsB-.5, cusB-.5)$.

On the other hand, suppose that G_1 and G_2 be two disjoint G_δ-sets. Since $G_1\cap G_2=\emptyset$. We have $G_2\subseteq G_1^c$. We set $f(x)=2$ for $x\in G_1^c$, $f(x)=\frac{1}{2}$ for $x\notin G_1^c$ and $g=\chi_{G_2}$.

Then since G_2 is a G_δ-set and G_1^c is a F_σ-set, we conclude that g is $clsB-.5$ and f is $cusB-.5$ and furthermore $g<f$. By hypothesis, there exists a Baire-.5 function h on X such that, $g<h<f$. Now we set $F_1=\{x\in X:h(x)\leq\frac{3}{4}\}$ and $F_2=\{x\in X:h(x)\geq 1\}$. Then F_1 and F_2 are two disjoint F_σ-sets contain G_1 and G_2, respectively. Hence $F_\sigma(G_1)\subseteq F_1$ and $F_\sigma(G_2)\subseteq F_2$ and consequently $F_\sigma(G_1)\cap F_\sigma(G_2)=\emptyset$. By Lemma 2.1, for every G of G_δ-set, the set $F_\sigma(G)$ is a G_δ-set.

Now suppose that $\{G_n\}$ is a decreasing sequence of G_δ-sets with empty intersection. We set $G_0=X$ and $f(x)=\frac{1}{n+1}$ for $x\in G_n\setminus G_{n+1}$. Since $\bigcap_{n=0}^{\infty}G_n=\emptyset$ and for every $n\in\mathbb{N}$ there exists $x\in G_n\setminus G_{n+1}, f$ is well-defined. Furthermore, for every $r\in\mathbb{R}$, if $r\leq 0$ then

$\{x \in X : f(x) < r\} = \emptyset$ is a G_δ-set and if $r > 0$ then by Archimedean property of $\mathbb{R}$, there exists $i \in \mathbb{N}$ such that $\frac{1}{i+1} \leq r$. Suppose that k is the least natural number with this property. Hence $\frac{1}{k} > r$. Now if $\frac{1}{k+1} < r$ then $\{x \in X : f(x) < r\} = G_k$ is a G_δ-set and if $\frac{1}{k+1} = r$ then $\{x \in X : f(x) < r\} = G_{k+1}$ is a G_δ-set. Hence f is a $cusB - .5$ on X. By setting $g = 0$, we have conclude that g is $clsB - .5$ on X and in addition $g < f$. By hypothesis there exists a Baire-.5 function h on X suvh that, $g < h < f$.

Set $F_n = \{x \in X : h(x) \leq \frac{1}{n+1}\}$. This set is a F_σ-set. But for every $x \in G_n$, we have $f(x) \leq \frac{1}{n+1}$ and since $g < h < f$ thus $h(x) < \frac{1}{n+1}$, this means that $x \in F_n$ and consequently $G_n \subseteq F_n$.

By definition of $F_n, \{F_n\}$ is a decreasing sequence of F_σ-sets and since $h > 0$, $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Thus the conditions holds. □

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References

[1] A. Al-Omari and M.S. Md Noorani. *Some properties of contra-b-continuous and almost contra-b-continuous functions*, European J. Pure. Appl. Math., **2**(2)(2009), 213–230.

[2] F. Brooks. *Indefinite cut sets for real functions*. Amer. Math. Monthly, **78**(1971), 1007–1010.

[3] M. Caldas and S. Jafari. *Some properties of contra-$\beta-$continuous functions*. Mem. Fac. Sci. Kochi. Univ., **22**(2001), 19–28.

[4] J. Dontchev. *The characterization of some peculiar topological space via $\alpha-$ and $\beta-$sets*. Acta Math. Hungar., **69**(1-2)(1995), 67–71.

[5] J. Dontchev. *Contra-continuous functions and strongly S-closed space*. Intrnat. J. Math. Math. Sci., **19**(2)(1996), 303–310.

[6] J. Dontchev and H. Maki. *On sg-closed sets and semi$-\lambda-$closed sets*. Questions Answers Gen. Topology, **15**(2)(1997), 259–266.

[7] E. Ekici. *On contra-continuity*. Annales Univ. Sci. Bodapest, **47**(2004), 127–137.

[8] E. Ekici. *New forms of contra-continuity*. Carpathian J. Math., **24**(1)(2008), 37–45.

[9] A.I. El-Magbrabi. *Some properties of contra-continuous mappings*. Int. J. General Topol., **3**(1-2)(2010), 55–64.

[10] M. Ganster and I. Reilly. *A decomposition of continuity*. Acta Math. Hungar., **56**(3-4)(1990), 299–301.

[11] S. Jafari and T. Noiri. *Contra-continuous function between topological spaces*. Iranian Int. J. Sci., **2**(2001), 153–167.

[12] S. Jafari and T. Noiri. *On contra-precontinuous functions*, Bull. Malaysian Math. Sc. Soc., **25**(2002), 115–128.

[13] M. Katětov. *On real-valued functions in topological spaces*. Fund. Math., **38**(1951), 85–91.

[14] M. Katětov. *Correction to, "On real-valued functions in topological spaces"*. Fund. Math., **40**(1953), 203–205.

[15] E. Lane. *Insertion of a continuous function*. Pacific J. Math., **66**(1976), 181–190.

[16] H. Maki. *Generalized* $\Lambda-$*sets and the associated closure operator*, The special Issue in commemoration of Prof. Kazuada IKEDA's Retirement, (1986), 139–146.

[17] S. N. Maheshwari and R. Prasad *On* R_{Os}*-spaces*. Portugal. Math., **34**(1975), 213–217.

[18] M. Mirmiran. *Insertion of a function belonging to a certain subclass of* $\mathbb{R}^X$. Bull. Iran. Math. Soc., Vol. **28**, No. 2 (2002), 19–27.

[19] M. Mrsevic. *On pairwise* R *and pairwise* R_1 *bitopological spaces*. Bull. Math. Soc. Sci. Math. R. S. Roumanie, **30**(1986), 141–145.

[20] A.A. Nasef. *Some properties of contra-continuous functions*. Chaos Solitons Fractals, **24**(2005), 471–477.

[21] M. Przemski. *A decomposition of continuity and* $\alpha-$*continuity*. Acta Math. Hungar., **61**(1-2)(1993), 93–98.

[22] H. Rosen. *Darboux Baire-.5 functions*. Proceedings of The American Mathematical Society, **110**(1)(1990), 285–286.

[23] M.H. Stone. *Boundedness properties in function-lattices*. Canad. J. Math.,**1**(1949), 176–189.