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Best proximity point results for Geraghty p-proximal cyclic quasi-contraction in uniform spaces

Resultado del punto más próximo para una casi-contracción cíclica p-proximal Geraghty en espacio uniformes

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Abstract

In this work, we develop Geraghty p-proximal cyclic quasi-contraction in uniform spaces. The existence and uniqueness of best proximity points for these contractions are proved. The main results, apart from the fact that they are new in literature, generalize several other similar results in literature. An illustrative example is given to validate the applicability of the results obtained.

Key words and phrases: Best proximity point, cyclic contraction, Geraghty p-proximal quasi-contraction, Geraghty p-proximal cyclic quasi-contraction, uniform spaces.

Resumen

En este trabajo se desarrolla la cuasi-contracción cíclica p-proximal de Geraghty en espacios uniformes, comprobándose la existencia y unicidad de los mejores puntos de proximidad para estas contracciones. Los principales resultados, además del hecho de que son nuevos en la literatura, generalizan varios otros resultados similares en la literatura. Se da un ejemplo ilustrativo para validar la aplicabilidad de los resultados obtenidos.

Palabras y frases clave: Mejor punto de proximidad, contracción cíclica, cuasi-contracción p-proximal de Geraghty, cuasi-contracción cíclica p-proximal de Geraghty, espacios uniformes.

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1 Introduction

Several problems can be modelled as equations of the form $Tx = x$, where T is a given self-mapping defined on a subset of a metric space, a normed linear space or some suitable spaces. However, if T is a non-self mapping from A to B, then the aforementioned equation does not necessarily admit a solution. In this case, it is appropriate to find an approximate solution x in A such that the error $d(x, Tx)$ is minimum, where d is the distance function. In view of the fact that $d(x, Tx)$ is at least $d(A, B)$, a best proximity point theorem guarantees the global minimization of $d(x, Tx)$ by the requirement that an approximate solution x satisfies the condition $d(x, Tx) = d(A, B)$. Such optimal approximate solutions are called best proximity points of the mapping T. Interestingly, best proximity theorems also serve as a natural generalization of fixed point theorems, for a best proximity point becomes a fixed point if the mapping under consideration is a self mapping, see [18], [20, 21, 26].

In [11], Eldred and Veeramani extended the cyclic contractive condition to the case when $A \cap B$ is empty and proved the existence of best proximity point. For other recent results on cyclic contractive conditions, see [2], [17] and [22]. Basha in [4] established some necessary and sufficient conditions for the existence of best proximity points for proximal contractions and gave some best proximity and convergence results in metric spaces. Mongkolkeha et al. in [19] generalized the results of Basha (cf. [4]) by introducing proximal cyclic contractions in metric spaces and proved existence results for best proximity point of the contraction. Thereafter, Jleli and Samet in [15] introduced the class of proximal quasi-contractive mappings and established best proximity point results for such mappings.

Geraghty in [12] extended the famous Banach Contraction Principle (cf. [3]) by introducing the generalized contraction mapping for self mapping using functions instead of constants. In 2012, Cabellero et al. in [7] generalized Geraghty (cf. [12]) by considering a non-self map and provided sufficient conditions for the existence of a unique best proximity point for Geraghty contractions. For other results on Geraghty contractions see [5, 7, 8, 9, 13, 18, 27].

Further improvement on Banach Contraction Principle include the use of uniform spaces which generalizes the metric space (cf. [10, 14, 23, 24, 26]). Weil in [28] was the first to introduce uniform spaces in terms of a family of pseudometrics and Bourbaki in [6] provided the definition of uniform structure in terms of entourages. Aamri and El Moutawakil in [1] gave some results on common fixed point of some contractive and expansive maps in uniform spaces and further introduced the definition of A-distance and E-distance. Most results in uniform spaces are of self mappings, however not many results of non-self mapping in uniform spaces exist in literature, (cf. [23]). More recent in 2018, a modified class of Hardy-Rogers p-proximal cyclic contraction in uniform spaces was introduced by Olisama et al. in [24] where the best proximity point results for this type of contraction was established.

Inspired by these, we introduce a class of Geraghty p-proximal cyclic quasi-contraction in uniform spaces and establish new best proximity point results for this type of contraction in uniform spaces. An illustrative example is given to demonstrate the usefulness of the established results.

2 Preliminaries

Here are some basic definitions and concepts relating to the main result of this paper.

Definition 2.1. (cf. [6]) A uniform space (X, Γ) is a non-empty set equipped with a uniform structure, which is a family Γ of subsets of Cartesian product $X \times X$, satisfying the following conditions:

(i) If $U \in \Gamma$, then U contains the diagonal $\Delta = \{(x, x) : x \in X\}$.

(ii) If $U \in \Gamma$, then $U^{-1} = \{(y, x) : (x, y) \in U\}$ is also in Γ.

(iii) If $U, V \in \Gamma$, then $U \cap V \in \Gamma$.

(iv) If $U \in \Gamma$, and $V \subseteq X \times X$ which contains U, then $V \in \Gamma$.

(v) If $U \in \Gamma$, then there exists $V \in \Gamma$ such that whenever (x, y) and (y, z) are in V, then (x, z) is in U.

Note that Γ is called the uniform structure or uniformity of X and its elements U and V are called neighbourhoods. A uniform structure Γ defines a unique topology $\tau(\Gamma)$ on X for which the neighbourhoods of $x \in X$ are the sets $V(x) = \{y \in X : (x, y) \in V\}, V \in \Gamma$.

Definition 2.2. (cf. [1]) Let (X, Γ) be a uniform space. A function $p : X \times X \to \Re^+$ is said to be an:

(a) A-distance if for any $V \in \Gamma$, there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.

(b) E-distance if p is an A-distance and $p(x, y) \leq p(x, z) + p(z, y)$, for $x, y, z \in X$.

Definition 2.3. (cf. [1]) Let (X, Γ) be a uniform space and p an A-distance on X.

(a) If $V \in \Gamma$, $(x, y) \in V$ and $(y, x) \in V$, then x and y are said to be V-close, and a sequence $(x_n)_{n=0}^{\infty} \in X$ is a Cauchy sequence for Γ if for any $V \in \Gamma$, there exists $N \geq 1$ such that x_n and x_m are V-close for $n, m \geq N$.

(b) A sequence in X is p-Cauchy if it satisfies the usual metric condition.

(c) X is S-complete if for every p-Cauchy sequence $(x_n)_{n=0}^{\infty} \in X$, there exists $x \in X$ such that $\lim\limits_{n\to\infty} p(x_n, x) = 0$. And X is p-Cauchy complete if for every p-Cauchy sequence $(x_n)_{n=0}^{\infty} \in X$, there exists $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$ with respect to $\tau(\Gamma)$.

(d) $f : X \times X$ is p-continuous if $\lim\limits_{n\to\infty} p(x_n, x) = 0$ implies $\lim\limits_{n\to\infty} p(T(x_n), T(x)) = 0$.

(e) X is said to be p-bounded if $\delta_p(X) = \sup\{p(x, y) : x, y \in X\} < \infty$.

Definition 2.4. (cf. [1]) A uniform space (X, Γ) is said to be Hausdorff if and only if the intersection of all the $V \in \Gamma$ reduces to the diagonal Δ of X. In other words, $(x, y) \in V$ for all $V \in \Gamma$ implies $x = y$.

Let A and B be non-empty subsets of a uniform space (X, Γ) such that p is an E-distance on X.

(i) $A_0 = \{x \in A : p(x, y) = p(A, B)$ for some $y \in B\}$.

(ii) $B_0 = \{y \in B : p(x, y) = p(A, B)$ for some $x \in A\}$.

(iii) Let $T : A \to B$, a point $x \in A$ is called a best proximity point if $p(x, Tx) = p(A, B)$ where $p(A, B) = \inf\{p(a, b) : a \in A,\ b \in B\}$.

The following lemma and definition are useful in this work.

Lemma 2.1. *(cf. [25]) Let (X, Γ) be a Hausdorff uniform space and p be an A-distance on X. Let $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty}$ be arbitrary sequences in X and $(\alpha_n)_{n=0}^{\infty}$, $(\beta_n)_{n=0}^{\infty}$ be sequences in $\Re^+$ converging to 0. Then, for $x, y, z \in X$, the following holds:*

(a) If $p(x_n, y) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ $\forall n \in N$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.

(b) If $p(x_n, y_n) = p(A, B)$ and $p(x_n, z_n) = p(A, B)$, then $y_n = z_n$, $\forall n \in N$ (cf. [24]).

(c) If $p(x_n, y_n) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ $\forall n \in N$, then, $(y_n)_{n=0}^{\infty}$ converges to z.

(d) If $p(x_n, x_m) \le \alpha_n$ $\forall m > n$, then $(x_n)_{n=0}^{\infty}$ is a p-Cauchy sequence in (X, Γ).

Definition 2.5. (cf. [24]) Let A, B be two non-empty subsets of a S-complete Hausdorff uniform space (X, Γ). Suppose $S : A \to B$ is a non self-mapping and $g : A \to A$ is an isometry, then S is said to preserve the isometric distance with respect to g if

$$p(S(g(x)), S(g(y))) = p(S(x), S(y))\ \forall x, y \in A. \tag{1}$$

3 Main Results

First of all, we introduce the following new concepts.

Let F be the family of all functions $\beta : [0, \infty) \to [0, 1)$ which satisfy the condition

$$\lim_{n\to\infty} \beta(t_n) = 1 \implies \lim_{n\to\infty} t_n = c, \quad c > 0.$$

Note that if $c = 0$ above, the equation reduces to the family of functions defined by Geraghty in [12].

Definition 3.1. Let (A, B) be a pair of non-empty subsets of an S-complete Hausdorff uniform space (X, Γ) such that p is an E-distance on X. A mapping $T : A \to B$ is said to be a Geraghty p-proximal quasi-contraction if there exists $\beta \in F$ such that for all $u, v, x, y \in A$

$$\begin{cases} p(u, T(x)) = p(A, B) \\ p(v, T(y)) = p(A, B) \end{cases} \implies p(u, v) \le \beta(M_T(x, y)) M_T(x, y), \tag{2}$$

where $M_T(x, y) = \max\{p(x, y);\ p(x, T(x));\ p(y, T(y));\ p(x, T(y));\ p(y, T(x))\}$.

Definition 3.2. Let (A, B) be a pair of non-empty subset of an S-complete Hausdorff uniform space (X, Γ) such that p is an E-distance on X. Suppose $T : A \to B$ and $G : B \to A$ are mappings. The pair (T, G) is said to be a Geraghty p-proximal cyclic quasi-contraction if there exists $\beta \in F$ such that for all $u, x \in A$, and $v, y \in B$

$$\begin{cases} p(u, T(x)) = p(A, B) \\ p(v, G(y)) = p(A, B) \end{cases} \implies p(u, v) \le \beta(M_T(x, y)) M_T(x, y) + (1 - \beta(M_T(x, y))) p(A, B), \tag{3}$$

where $M_T(x, y) = \max\{p(x, y);\ p(x, T(x));\ p(y, G(y));\ p(x, T(y));\ p(y, T(x))\}$.

Suppose (A, B) are a pair of non-empty subsets of a complete metric space i.e $\Gamma = \{(x, y) \in X^2 : d(x, y) < \epsilon\}$, $\beta(M_T(x, y)) = \alpha$, $\alpha \in [0, 1)$ and $M_T(x, y) = d(x, y)$, then (2) and (3) reduces to the proximal contraction and proximal cyclic contraction maps defined in [4] and [19], respectively.

Moreover, it is easy to see that a self mapping that is a Geraghty proximal quasi-contraction is a Geraghty quasi-contraction. But a non-self Geraghty p-proximal quasi-contraction is not necessarily a Geraghty quasi-contraction map in general sense.

Now, we state and prove the main result.

Theorem 3.1. *Let (X, Γ) be an Hausdorff uniform space and p an E-distance on X. Suppose (A, B) is a pair of non-empty closed subset of the p-bounded and S-complete space (X, Γ) such that A_0, $B_0 \neq \emptyset$. Let $T : A \to B$, $G : B \to A$ and $h : A \cup B \to A \cup B$ satisfy the following conditions:*

(i) T and G are Geraghty p-proximal quasi-contractions,

(ii) h is an isometry,

(iii) the pair (T,G) is a Geraghty p-proximal cyclic quasi-contraction,

(iv) $T(A_0) \subseteq B_0$, $G(B_0) \subseteq A_0$,

(v) $A_0 \subseteq h(A_0)$ and $B_0 \subseteq h(B_0)$.

Then there exist unique points $x \in A$ and $y \in B$ such that

$$p(h(x), T(x)) = p(h(y), G(y)) = p(x, y) = p(A, B).$$

Moreover, for any best proximity point $x_0 \in A_0$, the sequence $(x_n)_{n=0}^{\infty}$ defined by

$$p(h(x_{n+1}), T(x_n)) = p(A, B), \quad \forall n \geq 0$$

converges to the element $x \in A$.

Similarly, for any best proximity point $y_0 \in B_0$, the sequence $(y_n)_{n=0}^{\infty}$ defined by

$$p(h(y_{n+1}), G(y_n)) = p(A, B), \quad \forall n \geq 0$$

converges to the element $y \in B$.

Proof. Let $x_0 \in A_0$, since $A_0 \neq \emptyset$ and $T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that $p(x_1, T(x_0)) = p(A, B)$. Also, since $T(x_1) \in B_0$, there exists $x_2 \in A_0$ such that $p(x_2, T(x_1)) = p(A, B)$. Now, we obtain a sequence $(x_n)_{n=0}^{\infty} \subset A_0$ such that $p(x_{n+1}, T(x_n)) = p(A, B)$ $\forall n \in N$. Since T is a Geraghty p-proximal cyclic quasi-contraction, $\forall n \in N$ we have

$$\begin{aligned} p(x_{n+1}, T(x_n)) &= p(A, B), \\ p(x_n, T(x_{n-1})) &= p(A, B) \end{aligned} \tag{4}$$

and

$$p(x_{n+1}, x_n) \leq \beta(M_T(x_n, x_{n-1}))M_T(x_n, x_{n-1}) + (1 - \beta(M_T(x_n, x_{n-1}))p(A, B),$$

where

$$\begin{aligned} M_T(x_n, x_{n-1}) &= \max\{p(x_n, x_{n-1}); p(x_n, T(x_n)); p(x_{n-1}, T(x_{n-1})); \\ &\quad p(x_n, T(x_{n-1})); p(x_{n-1}, T(x_n))\} \\ &\leq \max\{p(x_n, x_{n-1}); [p(x_n, x_{n+1}) + p(x_{n+1}, T(x_n))]; \\ &\quad [p(x_{n-1}, x_n) + p(x_n, T(x_{n-1}))]; p(x_n, T(x_{n-1})); \\ &\quad [p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + p(x_{n+1}, T(x_n))]\} \\ &= p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + p(x_{n+1}, T(x_n)) \\ &= p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + p(A, B). \end{aligned}$$

Thus,

$$\begin{aligned} p(x_{n+1}, x_n) &\leq \beta(M_T(x_n, x_{n-1}))\,[p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + p(A, B)] \\ &\quad +(1 - \beta(M_T(x_n, x_{n-1}))p(A, B). \end{aligned}$$

Note that $p(x_{n+1}, x_n) \leq p(x_n, x_{n-1})$ for all $n \in \mathbb{N}$. Thus, the sequence $(p(x_{n+1}, x_n))_{n=0}^{\infty}$ is positive and decreasing. Since $\beta \in F$, by definition,

$$\lim_{n\to\infty} \beta(M_T(x_n, x_{n-1})) = 1 \implies \lim_{n\to\infty} M_T(x_n, x_{n-1}) = c.$$

Consequently,

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = p(A, B). \tag{5}$$

Next, we show that $(x_n)_{n=0}^{\infty}$ is a p-Cauchy sequence in the S-complete space X. That is,

$$\lim_{n\to\infty} p(x_n, x_m) = 0, \quad \text{for any} \quad n, m \in N.$$

Suppose, on the contrary, that $\epsilon = \lim\limits_{m,n\to\infty} p(x_n, x_m) > 0$. Since p is an E-distance, we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{m+1}) + p(x_{m+1}, x_m) \\ &\leq p(x_n, x_{n+1}) + \beta(M_T(x_n, x_m)M_T(x_n, x_m) + (1 - \beta(M_T(x_n, x_m)))p(A, B) \\ &\quad +p(x_{m+1}, x_m). \end{aligned}$$

But,

$$M_T(x_n, x_m) = \max\{p(x_n, x_m); p(x_n, T(x_n)); p(x_m, T(x_m)); p(x_n, T(x_m)); p(x_m, T(x_n))\}.$$

Taking limits as $m, n \to \infty$, $\lim\limits_{m,n\to\infty} p(x_n, x_m) = 0$. This is a contradiction. Therefore, the sequence $(x_n)_{n=0}^{\infty}$ is p-Cauchy in the S-complete space (X, Γ) whose limit is the unique best proximity point of T. Hence, $(x_n)_{n=0}^{\infty}$ converges to some element $x \in A$.

Similarly, since $G(B_0) \subseteq (A_0)$ and $B_0 \subseteq h(B_0)$, there exists a sequence $(y_n)_{n=0}^{\infty}$ such that it converges to some element $y \in B$. The pair (T, G) is a Geraghty p-proximal cyclic quasi-contraction, h is an isometry, by Lemma 2.1(b) and so,

$$p(h(x_{n+1}), T(x_n)) = p(h(y_{n+1}), G(y_n)) = p(A, B).$$

Now,

$$\begin{aligned}p(h(x_{n+1}), h(y_{n+1})) &= p(x_{n+1}, y_{n+1}) \\ &\leq \beta(M_T(x_n, y_n))M_T(x_n, y_n) + (1 - \beta(M_T(x_n, y_n)))p(A, B), \quad (6)\end{aligned}$$

where,

$$M_T(x_n, y_n) = \max\{p(x_n, y_n);\ p(x_n, T(x_n));\ p(y_n, G(y_n));\ p(x_n, G(y_n));\ p(y_n, T(x_n))\}.$$

Using Lemma 2.1(d) and taking limit as $n \to \infty$ in (4) yields:

$$p(x, y) = p(A, B). \quad (7)$$

Thus, $x \in A_0$ and $y \in B_0$. Since $T(A_0) \subseteq B_0$ and $G(B_0) \subseteq A_0$, there exist $h(x) \in A$ and $h(y) \in B$ such that

$$p(h(x), T(x)) = p(A, B) \quad (8)$$

and

$$p(h(y), G(y)) = p(A, B).$$

Thus, from (5) and (6), we get

$$p(x, y) = p(h(x), T(x)) = p(h(y), G(y)) = p(A, B).$$

Next, we prove the uniqueness of x and y. Suppose that there exist $x^* \in A$ and $y^* \in B$ with $x \neq x^*$ and $y \neq y^*$ such that

$$p(h(x^*), T(x^*)) = p(A, B), \quad (9)$$

and

$$p(h(y^*), G(y^*)) = p(A, B). \quad (10)$$

Since h is an isometry, and (T, G) is a Geraghty p-proximal cyclic quasi-contraction, using equations (6), (7) and Lemma 2.1(b) we have,

$$\begin{aligned}p(h(x), h(x^*)) = p(x, x^*) &\leq \beta(M_T(x, x^*))M_T(x, x^*) + (1 - \beta(M_T(x, x^*))p(A, B) \\ &< M_T(x, x^*) \quad (11)\end{aligned}$$

where

$$\begin{aligned}M_T(x, x^*) &= \max\{p(x, x^*);\ p(x,T(x));\ p(x^*, T(x^*));\ p(x, T(x^*));\ p(x^*, T(x)) \\ &= \max\{p(x, x^*);\ p(A, B)\}.\end{aligned}$$

Inequality (11) either gives $p(x, x^*) < p(x, x^*)$ or $p(x, x^*) < p(A, B)$, both of which are contradictions. Hence, $p(x, x^*) = 0$ and so $x^* = x$. Similarly, we show that $p(x^*, x) = 0$. But p is an E-distance, therefore

$$p(x^*, x^*) \leq p(x^*, x) + p(x, x^*).$$

Thus, $p(x^*, x^*) = 0$ and so $p(x, x^*) = p(x^*, x) = 0$. By Lemma 2.1(a), we conclude that $x^* = x$. Similarly, $y^* = y$ and the proof is complete. □

We are motivated by the example in [24] to support our obtained result in Theorem 3.1.

Example 3.1. *Consider the space* $X = \mathbb{R}$ *with Euclidean metric. Take the sets* $A = [-8, -2]$ *and* $B = [2, 8] \cup \{-12\}$*. Note that* $A_0 = -2,\ B_0 = 2$*.*

Now, let $T : A \to B$ *and* $G : B \to A$ *be defined by*

$$T(x) = \begin{cases} \dfrac{24}{x} & if \quad x < 0 \\ -\dfrac{13}{x} & if \quad x > 0 \end{cases}$$

and $G(y) = -\frac{20}{y}$, $y \neq 0$.

Suppose p is defined by:

$$p(x, y) = \begin{cases} \left|\dfrac{y}{2}\right| & if \quad x \leq y \\ 1 & if \quad x > y \end{cases} \quad (\textit{Note that } p(A, B) = 1).$$

Then p is an E-distance.

Taking $x_1 = -9,\ x_2 = -4,\ y_1 = 2$ *and* $y_2 = 20$,

$$d(x_1, T(x_2)) = d(y_1, G(y_2)) = d(A, B) = 3.$$

We show that the pair (T, G) *defined on a metric space, is not a Geraghty p-proximal cyclic quasi-contraction. By using (3),*

$$d(x_1, y_1) \leq \beta(M_T(x_2, y_2)M_T(x_2, y_2) + (1 - \beta(M_T(x_2, y_2)))d(A, B),$$

$$\begin{aligned} d(-9, 2) \leq\ & \beta(M_T(x_2, y_2)) \max\{d(-4, 20);\ d(-4, -6);\ d(20, -1);\ d(-4, -1);\ d(20, -6)\} \\ & + 3(1 - \beta(M_T(x_2, y_2))). \end{aligned}$$

Taking $\beta = \dfrac{1}{1+t}$*, we get*

$$11 > \frac{26}{27} + 3\left(1 - \frac{26}{27}\right),$$

a contradiction. Thus, (T, G) *is not a Geraghty p-proximal cyclic quasi-contraction on a metric space.*

Now, we consider the case where (T, G) *is defined on a uniform space. Clearly,* (T, G) *satisfies the Geraghty p-proximal cyclic quasi-contraction, for all* $x \in A$ *and* $y \in B$*, and* -2 *is the unique best proximity point of* T*, while* 2 *is the unique best proximity point of* G *and* $p(A, B) = 1$*.*

We now give the following corollaries to justify our case. Take $\beta(t) = k$, with $k \in [0, 1)$, then we have the following.

Corollary 3.1. *Let* (X, Γ) *be an Hausdorff uniform space and* p *an* E*-distance on* X*. Suppose* (A, B) *is a pair of non-empty closed subset of the p-bounded and S-complete space* (X, Γ) *such that* $A_0, B_0 \neq \emptyset$*. Let* $T : A \to B$*,* $G : B \to A$ *and* $h : A \cup B \to A \cup B$ *satisfy the following conditions:*

(i) T *and* G *are p-proximal quasi-contractions.*

(ii) h *is an isometry.*

(iii) The pair (T,G) is a p-proximal cyclic quasi-contraction.

(iv) $T(A_0) \subseteq B_0$, $G(B_0) \subseteq A_0$.

(v) $A_0 \subseteq h(A_0)$ *and* $B_0 \subseteq h(B_0)$.

Then there exist unique points $x \in A$ *and* $y \in B$ *such that*

$$p(h(x), T(x)) = p(h(y), G(y)) = p(x, y) = p(A, B).$$

Moreover, for any best proximity point $x_0 \in A_0$, *the sequence* $(x_n)_{n=0}^{\infty}$ *defined by*

$$p(h(x_{n+1}), T(x_n)) = p(A, B), \quad \forall n \geq 0$$

converges to the element x.

Similarly, for any best proximity point $y_0 \in B_0$, *the sequence* $(y_n)_{n=0}^{\infty}$ *defined by*

$$p(h(y_{n+1}), G(y_n)) = p(A, B), \quad \forall n \geq 0$$

converges to the element y.

If h becomes the identity mapping in Theorem 3.1 then we get the following.

Corollary 3.2. *Let* (X, Γ) *be an Hausdorff uniform space and* p *an* E*-distance on* X. *Suppose* (A, B) *is a pair of non-empty closed subset of the* p*-bounded and* S*-complete space* (X, Γ) *such that* A_0, $B_0 \neq \emptyset$. *Let* $T : A \to B$, $G : B \to A$ *and* $h : A \cup B \to A \cup B$ *satisfy the following conditions:*

(i) T *and* G *are Geraghty p-proximal quasi-contractions.*

(ii) The pair (T,G) is a Geraghty p-proximal cyclic quasi-contraction.

(iii) $T(A_0) \subseteq B_0$, $G(B_0) \subseteq A_0$.

Then there exist unique points $x \in A$ *and* $y \in B$ *such that*

$$p(x, T(x)) = p(y, G(y)) = p(x, y) = p(A, B).$$

If $M_T(x, y) = p(x, y)$ in Theorem 3.1, we obtain the following.

Corollary 3.3. *Let* (X, Γ) *be an Hausdorff uniform space and* p *an* E*-distance on* X. *Suppose* (A, B) *is a pair of non-empty closed subset of the* p*-bounded and* S*-complete space* (X, Γ) *such that* A_0, $B_0 \neq \emptyset$. *Let* $T : A \to B$, $G : B \to A$ *and* $h : A \cup B \to A \cup B$ *satisfy the following conditions:*

(i) T *and* G *are Geraghty p-proximal contractions.*

(ii) h *is an isometry.*

(iii) The pair (T,G) is a Geraghty p-proximal cyclic contraction.

(iv) $T(A_0) \subseteq B_0$, $G(B_0) \subseteq A_0$.

(v) $A_0 \subseteq h(A_0)$ *and* $B_0 \subseteq h(B_0)$.

Then there exist unique points $x \in A$ *and* $y \in B$ *such that*

$$p(h(x), T(x)) = p(h(y), G(y)) = p(x, y) = p(A, B).$$

Moreover, for any best proximity point $x_0 \in A_0$, *the sequence* $(x_n)_{n=0}^{\infty}$ *defined by*

$$p(h(x_{n+1}), T(x_n)) = p(A, B), \quad \forall n \geq 0$$

converges to the element x.

Similarly, for any best proximity point $y_0 \in B_0$, *the sequence* $(y_n)_{n=0}^{\infty}$ *defined by*

$$p(h(y_{n+1}), G(y_n)) = p(A, B), \quad \forall n \geq 0$$

converges to the element y.

Remark 3.1. Set $\Gamma = \{(x, y) \in X^2 : d(x, y) < \epsilon\}$ and suppose $M_T(x, y) = p(x, y)$, then Theorem 3.1 reduces to the result in [19]. In addition to that, if $\beta(t) = k$, $k \in [0, 1)$, then we obtain the result in [4]. Finally, if $\beta(t) = k$, $A = B$, h is the identity mapping and $\Gamma = \{(x, y) \in X^2 : d(x, y) < \epsilon\}$, then T has a unique fixed point and Theorem 3.1 reduces to the result in [9].

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