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# A note on some forms of continuity

*Una nota sobre algunas formas de continuidad*

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**Abstract**

New connections and characterizations of some classes of continuous functions are obtained. In particular, we characterize quasicontinuity and almost quasicontinuity in terms of weak types of open sets.

**Key words and phrases:** quasicontinuous; almost quasicontinuous; semicontinuous; precontinuous; $\alpha$-continuous; $b$-continuous; $\beta$-continuous; somewhat continuous; somewhat nearly continuous.

**Resumen**

Se obtienen nuevas conexiones y caracterizaciones de algunas clases de funciones continuas. En particular, caracterizamos la cuasi continuidad y casi cuasicontinuidad en términos de tipos débiles de conjuntos abiertos.

**Palabras y frases clave:** cuasicontinuo; casi cuasicontinuo; semicontinuo; precontinuo; $\alpha$-continuous; $b$-continuo; $\beta$-continuo; algo continuo; algo casi continuo.

## 1 Introduction

The role of continuity of functions is essential in developing theory in all branches of (pure) mathematics, especially in topology and analysis, for decades. Then various generalizations of continuity have been introduced. In 1932, Kempisty [13] defined the notion of quasicontinuity which has been of interest to many analysts and topologists, and there is a rich literature on these functions, see the survey article [18]. In 1958, Ptak [26] introduced nearly continuous functions to generalize the BanachSchauder Theorem. Levine [14] defined the concept of semicontinuity, in 1963, in terms of semiopen sets. Ten years later, Neubrunnova [19] showed that quasicontinuity and semicontinuity are similar. A big part of this work is motivated by that result. Gentry [12] has given a weaker class of quasicontinuity called somewhat continuity while studying the invariance of Baire spaces under mappings. Then $\alpha$-continuity was given by Njastad [20], which implies both nearly continuity and quasicontinuity. Mashhour et al. [16] have introduced the class of precontinuous functions which is equivalent to the class of nearly continuous functions. Abd-El-Monsef [1] studied $\beta$-continuous functions by using the notion of $\beta$-openness of sets. In 1990, Borsik [10] introduced an equivalent notion to $\beta$-continuity under the name of almost

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quasicontinuous functions. Almost quasicontinuity is weaker than both nearly continuity and quasicontinuity. In 1987, Piotrowski [23] defined a weak version of somewhat continuity called somewhat nearly continuity to generalize problems in separate versus joint continuity and in the Closed Graph Theorem. In 2009, Ameen [5] defined a subclass of quasicontinuous functions called *sc*-continuous. He showed that quasicontinuity and *sc*-continuity are identical on $T_1$-spaces. All such classes of functions mentioned earlier are weaker than the class of continuous functions except *sc*-continuity which is incomparable. Due to the importance of these classes of continuous functions, we present some more connections between these functions and give further characterizations.

## 2 Preliminaries and Auxiliary Materials

Throughout this paper, the letters $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively, stand for the set of natural, rational and real numbers. The word ”space” mean an arbitrary topological space. For a subset $A$ of a space $(X, \tau)$, the closure and interior of $A$ with respect to $X$ respectively are denoted by $\mathrm{Cl}_X(A)$ and $\mathrm{Int}_X(A)$ (or simply $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$).

**Definition 2.1.** A subset $A$ of a space X is said to be

(1) regular open if $A = \mathrm{Int}(\mathrm{Cl}(A))$,

(2) preopen [16] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$,

(3) semiopen [14] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$,

(4) *sc*-open [5] if $A$ is semiopen and union of closed sets,

(5) $\alpha$-open [20] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,

(6) $\gamma$-open [8] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$,

(7) $\beta$-open [1] or semipreopen [7] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,

(8) somewhat open (briefly *sw*-open) [23] if $\mathrm{Int}(A) \neq \emptyset$ or $A = \emptyset$,

(9) somewhat nearly open (briefly *swn*-open) [23] (for more details, see [4]) if $\mathrm{Int}(\mathrm{Cl}(A)) \neq \emptyset$ or $A = \emptyset$. The class of somewhat nearly open sets (except $\emptyset$) were studied under the name of somewhere dense sets in [2].

The complement of a regular open (resp. preopen, semiopen, *sc*-open, $\alpha$-open, $\beta$-open, $\gamma$-open, *sw*-open, *swn*-open) set is regular closed (resp. preclosed, semi-closed, *sc*-closed, $\alpha$-closed, $\beta$-closed, $\gamma$-closed, *sw*-closed, *swn*-closed).

The intersection of all preclosed (resp. semiclosed, $\alpha$-closed, $\beta$-closed, $\gamma$-closed) sets in $X$ containing $A$ is called the preclosure (resp. semi-closure, $\alpha$-closure, $\beta$-closure, $\gamma$-closure) of $A$, and is denoted by $\mathrm{Cl}_p(A)$ (resp. $\mathrm{Cl}_s(A)$, $\mathrm{Cl}_\alpha(A)$, $\mathrm{Cl}_\beta(A)$, $\mathrm{Cl}_\gamma(A)$).

The union of all preopen (resp. semiopen, $\alpha$-open, $\beta$-open, $\gamma$-open) sets in $X$ contained in $A$ is called the preinterior (resp. semi-interior, $\alpha$-interior, $\beta$-interior, $\gamma$-interior) of $A$, and is denoted by $\mathrm{Int}_p(A)$ (resp. $\mathrm{Int}_s(A)$, $\mathrm{Int}_\alpha(A)$, $\mathrm{Int}_\beta(A)$, $\mathrm{Int}_\gamma(A)$).

The family of all preopen (resp. semiopen, $\alpha$-open, $\gamma$-open, $\beta$-open) subsets of $X$ is denoted by $PO(X)$ (resp. $SO(X)$, $\alpha O(X)$, $\gamma O(X)$, $\beta O(X)$).

*Remark* 2.1. It is well-known that for a space $X$, $\tau \subseteq \alpha O(X) \subseteq PO(X) \cup SO(X) \subseteq \gamma O(X) \subseteq \beta O(X)$.

**Definition 2.2.** Let $X$ be a space and let $A \subseteq X$. A point $x \in X$ is said to be in the preclosure (resp. semi-closure, $\alpha$-closure, $\beta$-closure, $\gamma$-closure) of $A$ if $U \cap A \neq \phi$ for each preopen (resp. semiopen, $\alpha$-open, $\beta$-open, $\gamma$-open) set $U$ containing $x$.

**Lemma 2.1.** *Let $A$ be a subset of a space $X$.*

*(i) $A$ is semiopen if and only if* $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(A))$.

*(ii) $A$ is $\beta$-open if and only if* $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

*Proof.* $(i)$ If $A$ is semiopen, then $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$ and so $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. For other side of inclusion, we always have $\mathrm{Int}(A) \subseteq A$. Therefore $\mathrm{Cl}(\mathrm{Int}(A)) \subseteq \mathrm{Cl}(A)$. Thus $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(A))$.

Conversely, assume that $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(A))$, but $A \subseteq \mathrm{Cl}(A)$ always, so $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. Hence $A$ is semiopen.

$(ii)$ Theorem 2.4 in [7]. □

**Lemma 2.2.** *Let $A$ be a nonempty subset of a space $X$.*

*(i) If $A$ is semiopen, then* $\mathrm{Int}(A) \neq \emptyset$.

*(ii) If $A$ is $\beta$-open, then* $\mathrm{Int}(\mathrm{Cl}(A)) \neq \emptyset$.

*Proof.* $(i)$ Suppose otherwise that if $A$ is a semiopen set such that $\mathrm{Int}(A) = \emptyset$, by Lemma 2.1 $(i)$, $\mathrm{Cl}(A) = \emptyset$ which implies that $A = \emptyset$. Contradiction!

$(ii)$ Similar to $(i)$. □

At this place, perhaps a connection among the classes of open sets (defined above excluding $\gamma$-open as we have only used in Theorem 4.3) is needed.

$$\begin{tikzcd}
\text{open set} \arrow[d] & sc\text{-open set} \arrow[l, "/" marking] \arrow[d] & \text{regular closed set} \arrow[l] \arrow[d] \\
\alpha\text{-open set} \arrow[r] \arrow[d] & \text{semiopen set} \arrow[r] \arrow[d] & sw\text{-open set} \arrow[d] \\
\text{preopen set} \arrow[r] & \beta\text{-open set} \arrow[r] & swn\text{-open set}
\end{tikzcd}$$

Diagram I

In general, none of these implications can be replaced by equivalence as shown below:

**Example 2.1.** *Consider $\mathbb{R}$ with the usual topology. Let $A = \mathbb{R} \backslash \{\frac{1}{n}\}_{n \in \mathbb{N}}$. Obviously $A$ is $\alpha$-open but not open. If $B = [0,1]$, $B$ is semiopen but not $\alpha$-open. If $C = \mathbb{Q}$, $C$ is preopen but not $\alpha$-open. Let $D = [0,1] \cup ((1,2) \cap \mathbb{Q})$. Then $D$ is both $\beta$-open and sw-open but neither preopen nor semiopen ([8, Example 1]). If $E = [0,1) \cap \mathbb{Q}$, then $E$ is swn-open but not sw-open. Let $F = \mathcal{C} \cup [2,3]$, where $\mathcal{C}$ is the Cantor set. Then $F$ is swn-open but not $\beta$-open. Let $G = (0,1] = \bigcup_{n=1}^{\infty} [\frac{1}{n}, 1]$. So $G$ is sc-open but neither open nor regular closed.*

**Example 2.2.** *[5, Example 2.2.3] Consider* $X = \{a, b, c\}$ *with the topology* $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. *The set* $\{a\}$ *is semiopen but not sc-open.*

**Lemma 2.3.** *[7, Theorems 3.13, 3.14 & 3.22][8, Proposition 2.6] For a subset* $A$ *of a space* $X$, *we have*

*(i)* $\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Cl}(\mathrm{Int}_s(A)) = \mathrm{Cl}_\alpha(\mathrm{Int}_\alpha(A)) = \mathrm{Cl}(\mathrm{Int}_\alpha(A)) = \mathrm{Cl}_\alpha(\mathrm{Int}(A))$,

*(ii)* $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) = \mathrm{Cl}(\mathrm{Int}_p(A)) = \mathrm{Cl}(\mathrm{Int}_\gamma(A)) = \mathrm{Cl}(\mathrm{Int}_\beta(A))$,

*(iii)* $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) = \mathrm{Cl}_s(\mathrm{Int}(A)) = \mathrm{Cl}_\gamma(\mathrm{Int}(A)) = \mathrm{Cl}_\beta(\mathrm{Int}(A))$,

**Lemma 2.4.** *Let* $A, B$ *be subsets of* $X$. *If* $A$ *is open and* $B$ *is* $\alpha$*-open (resp. preopen, semiopen,* $\beta$*-open), then* $A \cap B$ *is* $\alpha$*-open (resp. preopen, semiopen,* $\beta$*-open) in* $X$.

*Proof.* Proposition 2 in [20] (resp. Lemma 4.1 in [24], Lemma 1 in [21], Theorem 2.7 in [1]). □

**Lemma 2.5.** *Let* $A, B$ *be subsets of a space* $X$.

*(i) If* $A$ *is semiopen and* $B$ *is preopen, then* $A \cap B$ *is semiopen in* $B$, *[17, Lemma 1.1].*

*(ii) If* $A$ *is semiopen and* $B$ *is preopen, then* $A \cap B$ *is preopen in* $A$, *[17, Lemma 2.1].*

**Lemma 2.6.** *Let* $Y$ *be a subspace of a space* $X$ *and let* $A \subset Y$.

*(i) If* $Y$ *is semiopen in* $X$, *then* $A$ *is semiopen in* $Y$ *if and only if* $A$ *is semiopen in* $X$.

*(ii) If* $Y$ *is* $\beta$*-open in* $X$, *then* $A$ *is* $\beta$*-open in* $Y$ *if and only if* $A$ *is* $\beta$*-open in* $X$.

*(iii) If* $Y$ *is semiopen in* $X$, *then* $A$ *is swn-open in* $Y$ *if and only if it is swn-open in* $X$.

*Proof.* $(i)$ [15, Theorem 2.4].

$(ii)$ The fist direction is proved in [1, Theorem 2.7]. The converse is can be followed from [14, Theorem 6] and from the fact that $A$ is $\beta$-open if and only if there exist a preopen open $U$ such that $U \subseteq A \subseteq \mathrm{Cl}(U)$.

$(iii)$ [4, Theorem 3.14]. □

**Lemma 2.7.** *[20, Proposition 1] Let* $X$ *be a space. A subset* $A$ *of* $X$ *is* $\alpha$*-open if and only if* $A \cap B$ *is semiopen for each semiopen subset* $B$ *of* $X$.

In a similar way, we prove the following:

**Lemma 2.8.** *Let* $X$ *be a space. A subset* $A$ *of* $X$ *is preopen if and only if* $A \cap B$ *is* $\beta$*-open for each semiopen subset* $B$ *of* $X$.

*Proof.* Given subsets $A, B \subseteq X$ such that $A$ is preopen and $B$ is semiopen. Let $x \in A \cap B$ and let $U$ be an open set containing $x$. Since $x \in \mathrm{Int}(\mathrm{Cl}(A))$, then $U \cap \mathrm{Int}(\mathrm{Cl}(A))$ is also an open set containing $x$. Set $V = U \cap \mathrm{Int}(\mathrm{Cl}(A))$. But $x \in \mathrm{Cl}(\mathrm{Int}(B))$, so

$$U \cap \mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Int}(B) = V \cap \mathrm{Int}(B) \neq \emptyset.$$

This implies that

$$A \cap B \subseteq \mathrm{Cl}\left[\mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Int}(B)\right] = \mathrm{Cl}\left[\mathrm{Int}\left[\mathrm{Cl}(A) \cap \mathrm{Int}(B)\right]\right],$$

and therefore,

$$A \cap B \subseteq \mathrm{Cl}\left[\mathrm{Int}\left[\mathrm{Cl}(A) \cap \mathrm{Int}(B)\right]\right] \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A \cap B))).$$

Hence $A \cap B$ is $\beta$-open.

Conversely, assume that $A \cap B$ is $\beta$-open for each semiopen set $B$ in $X$. We need to show that $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$. Suppose contrary that there exists $x \in X$ such that $x \in A$ and $x \notin \mathrm{Int}(\mathrm{Cl}(A))$. Then $x \in \mathrm{Cl}(\mathrm{Int}(A^c))$ and obviously $\mathrm{Int}(A^c) \cup \{x\}$ is semiopen. By assumption, $A \cap (\mathrm{Int}(A^c) \cup \{x\})$ is $\beta$-open. But $A \cap (\mathrm{Int}(A^c) \cup \{x\}) = \{x\}$. By Lemma 2.2 (*ii*) and [6, Lemma 2.1], $\{x\}$ is preopen. This implies $x \in \mathrm{Int}(\mathrm{Cl}(A))$, which contradicts our assumption. Therefore, if $x \in A$, then $x \in \mathrm{Int}(\mathrm{Cl}(A))$ and so $A$ is preopen. □

**Lemma 2.9.** *[4, Proposition 3.16] Let $X$ be a space. A subset $A$ of $X$ is $\beta$-open if and only if $A \cap U$ is swn-open for each open set $U$ in $X$.*

**Lemma 2.10.** *Let $X$ be a space. A subset $A$ of $X$ is semiopen if and only if $A \cap U$ is $sw$-open for each open set $U$ in $X$.*

*Proof.* Since each semiopen set is $sw$-open and the intersection of a semiopen set with an open set is semiopen, by Lemma 2.4, so the first part follows.

Conversely, let $x \in A$ and assume that $A \cap U$ is $sw$-open for each open set $U$ in $X$. That is $\mathrm{Int}(A \cap U) \neq \emptyset$. But $\emptyset \neq \mathrm{Int}(A \cap U) = \mathrm{Int}(A) \cap \mathrm{Int}(U) = \mathrm{Int}(A) \cap U$, which implies that $x \in \mathrm{Cl}(\mathrm{Int}(A))$ and so $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. This proves that $A$ is semiopen. □

**Lemma 2.11.** *Let $X$ be a space. A subset $A$ of $X$ is $\alpha$-open if and only if $A \cap U$ is $sw$-open for each $\alpha$-open set $U$ in $X$.*

*Proof.* Since the intersection of two $\alpha$-open sets is $\alpha$-open and each $\alpha$-open set is $sw$-open, so the first part is proved.

Conversely, let $x \in A$ and assume that $A \cap U$ is $sw$-open for each $\alpha$-open set $U$ in $X$. That is $\mathrm{Int}(A \cap U) \neq \emptyset$. But $\emptyset \neq \mathrm{Int}(A \cap U) = \mathrm{Int}(A) \cap \mathrm{Int}(U) \subseteq \mathrm{Int}(A) \cap \mathrm{Int}(\mathrm{Cl}(U)) = \mathrm{Int}(A) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))$, which implies that $\mathrm{Int}(A) \cap Cl_\beta(U) \neq \emptyset$ and therefore $x \in \mathrm{Cl}_\beta(\mathrm{Int}(A) \cap U) \subseteq \mathrm{Cl}_\beta(\mathrm{Int}(A))$. By Lemma 2.3 (*iii*), $\mathrm{Cl}_\beta(\mathrm{Int}(A)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ and so $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. This proves that $A$ is $\alpha$-open. □

**Lemma 2.12.** *Let $X$ be a space. The following are equivalent:*

*(i)* *each preopen subset of $X$ is $\alpha$-open,*

*(ii)* *each $\beta$-open subset of $X$ is semiopen,*

*(iii)* *each preopen subset of $X$ is semiopen,*

*(iv)* *each dense subset of $X$ is semiopen,*

*(v)* *each dense subset of $X$ has an interior dense,*

*(vi)* *each co-dense subset of $X$ is nowhere dense,*

*(vii)* *each swn-open subset of $X$ is $sw$-open,*

*(viii)* *each subset of $X$ has a nowhere dense boundary.*

*Proof. (i) ⇒ (ii):* Let $A$ be a $\beta$-open set in $X$. Then $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. By $(i)$, $\mathrm{Int}(\mathrm{Cl}(A)) \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. Therefore, $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) \subseteq \mathrm{Cl}[\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))] = \mathrm{Cl}(\mathrm{Int}(A))$. Hence $A$ is semiopen.

The implications "*(ii) ⇒ (iii)*" and "*(iii) ⇒ (iv)*" are clear as each dense is preopen and each preopen is $\beta$-open.

*(iv) ⇒ (v):* Let $D$ be a dense subset of $X$. By Lemma 2.1 $(i)$, $X = \mathrm{Cl}(D) = \mathrm{Cl}(\mathrm{Int}(D))$. Thus $\mathrm{Int}(D)$ is dense.

$(v) \Leftrightarrow (vi)$: Let $A$ be co-dense. Then $\mathrm{Int}(A) = \emptyset \Longleftrightarrow \mathrm{Cl}(X \setminus A) = X$. By $(v)$, $\mathrm{Cl}(\mathrm{Int}(X \setminus A)) = X \Longleftrightarrow \mathrm{Int}(\mathrm{Cl}(A)) = \emptyset$. Hence $A$ is nowhere dense.

*(vi) ⇔ (vii):* Let $A$ be an $swn$-open set in $X$. Suppose $A$ is not $sw$-open. That is, $A$ is co-dense. By $(vi)$, $\mathrm{Int}(\mathrm{Cl}(A)) = \emptyset$. Contradiction that assumption that $A$ is $swn$-open. The other way is similar.

*(v) ⇔ (viii):* Let $A$ be a subset of $X$. Then $X = \mathrm{Cl}(A) \cup (X \setminus \mathrm{Cl}(A)) = \mathrm{Cl}(A) \cup \mathrm{Int}(X \setminus A) \subseteq \mathrm{Cl}[A \cup \mathrm{Int}(X \setminus A)]$. This implies that $A \cup \mathrm{Int}(X \setminus A)$ is dense in $X$. By the same, we can conclude that $\mathrm{Int}(A) \cup X \setminus A$ is also dense in $X$. By $(v)$, both $\mathrm{Int}[A \cup \mathrm{Int}(X \setminus A)]$ and $\mathrm{Int}[\mathrm{Int}(A) \cup X \setminus A]$ are (open) dense. Now,

$$\begin{aligned}
\mathrm{Int}[A \cup \mathrm{Int}(X \setminus A)] \bigcap \mathrm{Int}[\mathrm{Int}(A) \cup (X \setminus A)] &= \mathrm{Int}[A \cup \mathrm{Int}(X \setminus A) \bigcap \mathrm{Int}(A) \cup (X \setminus A)] \\
&= \mathrm{Int}[\mathrm{Int}(A) \bigcup \mathrm{Int}(X \setminus A)] \\
&= X \setminus \partial(A),
\end{aligned}$$

where $\partial(A)$ means the topological boundary of $A$. Since the intersection of two open dense is dense, so $X \setminus \partial(A)$ is open dense. Thus $\partial(A)$ is nowhere dense.

*(viii) ⇔ (i):* Let $A$ be preopen. That is $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$. By $(viii)$, $\emptyset = \mathrm{Int}(\mathrm{Cl}(\partial(A))) = \mathrm{Int}(\partial(A)) = \mathrm{Int}(\mathrm{Cl}(A)) \setminus \mathrm{Cl}(\mathrm{Int}(A))$. It follows that $A \subseteq \mathrm{Int}(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(\mathrm{Int}(A))$ and so $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(A))$. Since $A \subseteq \mathrm{Int}[\mathrm{Cl}(A)] = \mathrm{Int}[\mathrm{Cl}(\mathrm{Int}(A))]$. This proves that $A$ is $\alpha$-open. □

# 3 Relationships and properties

This section is devoted to some properties of the following classes of continuous functions and their relationships.

**Definition 3.1.** A function $f$ from a space $X$ to a space $Y$ is called

(1) $rc$-continuous [11], if the inverse image of each open set in $Y$ is regular closed in $X$,

(2) $sc$-continuous [5], if the inverse image of each open set in $Y$ is $sc$-open in $X$,

(3) semicontinuous [14], if the inverse image of each open set in $Y$ is semiopen in $X$,

(4) nearly continuous [26], or precontinuous [16], if the inverse image of each open set in $Y$ is preopen in $X$,

(5) $\alpha$-continuous [20], if the inverse image of each open set in $Y$ is $\alpha$-open in $X$,

(6) $\beta$-continuous [1], if the inverse image of each open set in $Y$ is $\beta$-open in $X$,

(7) somewhat continuous [12] (briefly $sw$-continuous), if the inverse image of each open set in $Y$ is $sw$-open in $X$,

(8) somewhat nearly continuous [23, 4] (briefly *swn*-continuous), if the inverse image of each open set in $Y$ is *swn*-open in $X$,

(9) quasicontinuous [13], if for each $x \in X$, each open set $G$ containing $f(x)$ and each open $U$ containing $x$, there exists a nonempty open set $V$ with $V \subseteq U$ such that $f(V) \subseteq G$,

(10) almost quasicontinuous [10], if for each $x \in X$, each open set $G$ containing $f(x)$ and each open $U$ containing $x$, $f^{-1}(G) \cap U$ is not nowhere dense.

*Remark* 3.1. (*i*) It is proved in [25] that almost quasicontinuity and $\beta$-continuity are equivalent (see also [9, Theorem 1]). An easier proof can be followed from the definition of almost quasicontinuity and Lemma 2.9.
(*ii*) The equivalence of semicontinuity and quasicontinuity is given in [19, Theorem 1.1].
(*iii*) somewhat nearly continuous functions coincide with surjective SD-continuous in [3].

The following diagram shows the relationship between above functions, which is an enlargement of the Diagram I given in [23]:

$$\begin{tikzcd}
\text{continuous} \arrow[d] & sc\text{-continuous} \arrow[l, "/" marking] \arrow[d] & rc\text{-continuous} \arrow[l] \arrow[d] \\
\alpha\text{-continuous} \arrow[r] \arrow[d] & \text{quasicontinuous} \arrow[r] \arrow[d] & sw\text{-continuous} \arrow[d] \\
\text{nearly continuous} \arrow[r] & \text{almost quasicontinuous} \arrow[r] & swn\text{-continuous}
\end{tikzcd}$$

Diagram II

In general, none of the implications is reversible. Examples 5.2-5.3 in [4] show that the existence of *swn*-continuous functions that are not almost quasicontinuous or *sw*-continuous. Counterexamples for other cases are available in the literature.

**Theorem 3.1.** *Let $X, Y$ be spaces such that $X$ is $T_1$. A function $f : X \to Y$ is $\alpha$-continuous if and only if it is both sc-continuous and nearly continuous.*

*Proof.* Proposition 2.2.10 in [5] and Theorem 3.2 in [22]. □

**Theorem 3.2.** *[9, Proposition 1] Let $X, Y$ be spaces. A function $f : X \to Y$ is almost quasicontinuous if and only if $f|_U$ is swn-continuous for each open subset $U$ of $X$.*

**Theorem 3.3.** *[4, Theorem 5.7] Let $X, Y$ be spaces. A function $f : X \to Y$ is nearly continuous if and only if $f|_U$ is swn-continuous for each $\alpha$-open subset $U$ of $X$.*

Similar to the above results, we prove the following:

**Theorem 3.4.** *Let $X, Y$ be spaces. A function $f : X \to Y$ is quasicontinuous if and only if $f|_U$ is sw-continuous for each open subset $U$ of $X$.*

*Proof.* The first part follows from [17, Theorem 1.3], which implies that each quasicontinuous restricted to an open set is again quasicontinuous and hence $sw$-continuous.

Conversely, suppose that $f|_U$ is $sw$-continuous for each open subset $U$ of $X$. Let $H$ be an open in $Y$. Then $f^{-1}|_U(H) = f^{-1}(H) \cap U$ is $sw$-open in $U$. Since $U$ is an open subset of $X$, clearly, $f^{-1}(H) \cap U$ is $sw$-open in $X$ and so, by Lemma 2.10, $f^{-1}(H)$ is semiopen in $X$. Thus $f$ is quasicontinuous. □

**Theorem 3.5.** *Let $X, Y$ be spaces. A function $f : X \to Y$ is $\alpha$-continuous if and only if $f|_U$ is $sw$-continuous for each $\alpha$-open subset $U$ of $X$.*

*Proof.* Similar steps given in the proof of the above theorem and Lemma 2.11. □

**Lemma 3.1.** *Let $A, B$ be subsets of a space $X$. If $A$ is semiopen and $B$ is $\alpha$-open, $A \cap B$ is $\alpha$-open in $A$.*

*Proof.* Given the sets $A, B$, then

$$\begin{aligned} A \cap B &\subseteq A \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B))) \\ &\subseteq \mathrm{Int}_A[A \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))] \\ &\subseteq \mathrm{Int}_A[\mathrm{Cl}(\mathrm{Int}(A)) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))] \\ &\subseteq \mathrm{Int}_A[\mathrm{Cl}[\mathrm{Int}(A) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))]] \\ &\subseteq \mathrm{Int}_A(\mathrm{Cl}(\mathrm{Int}(A \cap B))) \\ &\subseteq \mathrm{Int}_A(\mathrm{Cl}(\mathrm{Int}_A(A \cap B))). \end{aligned}$$

Since $\mathrm{Int}_A(\mathrm{Cl}(\mathrm{Int}_A(A \cap B)))$ is an open set in $A$, so $\mathrm{Int}_A(\mathrm{Cl}(\mathrm{Int}_A(A \cap B))) \cap A = \mathrm{Int}_A(\mathrm{Cl}(\mathrm{Int}_A(A \cap B)) \cap A)$, and hence $A \cap B \subseteq \mathrm{Int}_A(\mathrm{Cl}(\mathrm{Int}_A(A \cap B)) \cap A) = \mathrm{Int}_A(\mathrm{Cl}_A(\mathrm{Int}_A(A \cap B)))$. This shows that $A \cap B$ is $\alpha$-open in $A$. □

**Theorem 3.6.** *Let $X, Y$ be spaces. A function $f : X \to Y$ is $\alpha$-continuous if and only if $f|_U$ is quasicontinuous for each semiopen subset $U \subseteq X$.*

*Proof.* Assume that $f$ is $\alpha$-continuous. Let $H$ be an open subset of $Y$ and let $U$ be a semiopen subset of $X$. By assumption $f^{-1}(H)$ is $\alpha$-open in $X$. By Lemma 3.1, $f^{-1}(H) \cap U$ is $\alpha$-open in $U$ and thus, by Diagram I, $f^{-1}(H) \cap U$ is a semiopen subset of $U$. Hence, $f|_U$ is quasicontinuous.

Conversely, suppose that $f|_U$ is quasicontinuous for each semiopen subset $U$ of $X$. Let $H$ be an open set in $Y$. Then $f^{-1}|_U(H) = f^{-1}(H) \cap U$ is semiopen in $U$. Since $U$ is semiopen in $X$, by Lemma 2.6 ($i$), $f^{-1}(H) \cap U$ is semiopen in $X$ for each semiopen $U$ and thus, by Lemma 2.7, $f^{-1}(H)$ is $\alpha$-open in $X$. Thus $f$ is $\alpha$-continuous. □

**Theorem 3.7.** *Let $X, Y$ be spaces. A function $f : X \to Y$ is nearly continuous if and only if $f|_U$ is almost quasicontinuous for each semiopen subset $U \subseteq X$.*

*Proof.* Suppose that $f$ is nearly continuous. Let $H$ be an open subset of $Y$ and let $U$ be a semiopen subset of $X$. By hypothesis $f^{-1}(H)$ is preopen in $X$. By Lemma 2.5 ($ii$), $f^{-1}(H) \cap U$ is preopen in $U$ and thus, by Diagram I, $f^{-1}(H) \cap U$ is $\beta$-open in $U$. Therefore, $f|_U$ is almost quasicontinuous.

Conversely, suppose that $f|_U$ is almost quasicontinuous for each semiopen subset $U$ of $X$. Let $H$ be an open set in $Y$. Then $f^{-1}|_U(H) = f^{-1}(H) \cap U$ is a $\beta$-open subset of $U$. Since $U$ is semiopen in $X$ and each semiopen is $\beta$-open, by Lemma 2.6 ($ii$), $f^{-1}(H) \cap U$ is $\beta$-open in

$X$ for each semiopen $U$ and thus, by Lemma 2.8, $f^{-1}(H)$ is preopen in $X$. Thus $f$ is nearly continuous. $\square$

**Theorem 3.8.** *For a function $f : X \to Y$, the following are equivalent:*

*(i) each nearly continuous function is $\alpha$-continuous,*

*(ii) each almost quasicontinuous function is quasicontinuous,*

*(iii) each nearly continuous function is quasicontinuous,*

*(iv) each swn-continuous function is sw-continuous.*

*Proof.* Apply Lemma 2.12 $\square$

# 4 Characterizations

**Theorem 4.1.** *Let $X, Y$ be spaces. For a function $f : X \to Y$, the following are equivalent:*

*(1) $f$ is quasicontinuous;*

*(2) For each $x \in X$, each open set $G$ containing $f(x)$ and each open $U$ containing $x$, $f^{-1}(G) \cap U$ is sw-open;*

*(3) For each $x \in X$, each open set $G$ containing $f(x)$ and each $\alpha$-open $U$ containing $x$, there exists a nonempty open set $V$ with $V \subseteq U$ such that $f(V) \subseteq G$;*

*(4) For each $x \in X$, each open set $G$ containing $f(x)$ and each $\alpha$-open $U$ containing $x$, there exists a nonempty $\alpha$-open set $V$ with $V \subseteq U$ such that $f(V) \subseteq G$;*

*(5) For each $x \in X$, each open set $G$ containing $f(x)$ and each open $U$ containing $x$, there exists a nonempty $\alpha$-open set $V$ with $V \subseteq U$ such that $f(V) \subseteq G$;*

*(6) For each $x \in X$, each open set $G$ containing $f(x)$ and each open $U$ containing $x$, there exists a nonempty semiopen set $V$ with $V \subseteq U$ such that $f(V) \subseteq G$.*

*Proof. (1)⇒(2):* Let $G$ be an open set containing $f(x)$ and let $U$ be any open containing $x$. By (1), there is a nonempty open set $V$ with $V \subseteq U$ such that $f(V) \subseteq G$. Therefore $V \subseteq f^{-1}(G)$ and so $V \subseteq \text{Int}(f^{-1}(G))$. Thus $\emptyset \neq V = V \cap U \subseteq \text{Int}(f^{-1}(G)) \cap U = \text{Int}(f^{-1}(G) \cap U)$, which implies that $f^{-1}(G) \cap U$ is *sw*-open.

*(2)⇒(3):* Let $G$ be an open set in $Y$ containing $f(x)$ and let $U$ be an $\alpha$-open set in $X$ containing $x$. Since each $\alpha$-open set is semiopen, by Lemma 2.2 ($i$), $\text{Int}(U)$ is a nonempty open set. By (2) $\text{Int}(f^{-1}(G) \cap \text{Int}(U)) = \text{Int}(f^{-1}(G)) \cap \text{Int}(U) \neq \emptyset$. Set $V = \text{Int}(f^{-1}(G)) \cap \text{Int}(U)$. Clearly, $V$ is a nonempty open set $U$ and

$$f(V) \subseteq f(\text{Int}(f^{-1}(G)) \cap \text{Int}(U)) \subseteq f(f^{-1}(G)) \subseteq G.$$

This proves (3).

The implications "*(3)⇒(4)*", "*(4)⇒(5)*" and "*(5)⇒(6)*" are clear from the Diagram $I$.

*(6)⇒(1):* Let $G$ be an open set containing $f(x)$. By (6), for each open set $U$ containing $x$, there is a nonempty semiopen set $V$ with $V \subseteq U$ such that $f(V) \subseteq G$. Therefore $V \subseteq f^{-1}(G)$ and so $V \subseteq \text{Int}_s(f^{-1}(G))$. Thus $\emptyset \neq V = V \cap U \subseteq \text{Int}_s(f^{-1}(G)) \cap U$, which implies that $\text{Int}_s(f^{-1}(G)) \cap U \neq \emptyset$ for each open $U$ containing $x$. Hence $x \in \text{Cl}(\text{Int}_s(f^{-1}(G)))$. By Lemma 2.3 ($i$), $x \in f^{-1}(G) \subseteq \text{Cl}(\text{Int}(f^{-1}(G)))$. As $x$ was taken arbitrarily, so $f$ is quasicontinuous. $\square$

The proofs of the following theorems are quite similar to the proof of Theorem 4.1. But for the sake of completeness, we provide them.

**Theorem 4.2.** *Let $X, Y$ be spaces. For a function $f : X \to Y$, the following are equivalent:*

*(1) $f$ is $\alpha$-continuous;*

*(2) For each $x \in X$, each open set $G$ containing $f(x)$ and each semiopen $U$ containing $x$, there exist a nonempty open set $V$ with $V \subseteq U$ such that $f(V) \subseteq G$.*

*Proof. (1)⇒(2):* Let $G$ be an open set containing $f(x)$ and let $U$ be a semiopen containing $x$. By (1) and Lemma 2.3 ($iii$), $x \in f^{-1}(G) \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(G)))) = \mathrm{Cl}_s(\mathrm{Int}(f^{-1}(G)))$. This implies that $\mathrm{Int}(f^{-1}(G)) \cap B \neq \emptyset$ for each semiopen $B$ containing $x$ and so $\mathrm{Int}(f^{-1}(G)) \cap U \neq \emptyset$. If $W = \mathrm{Int}(f^{-1}(G)) \cap U$, by Lemma 2.4, $W$ is a nonempty semiopen set. Set $V = \mathrm{Int}(W)$. By Lemma 2.2 ($i$), $V$ is nonempty open and

$$f(V) \subseteq f(\mathrm{Int}(f^{-1}(G)) \cap U) \subseteq f(f^{-1}(G)) \subseteq G.$$

This completes the proof of (2).

*(2)⇒(1):* Let $x \in X$ and let $G$ be an open set containing $f(x)$. By (2), for each semiopen set $U$ containing $x$, there is a nonempty open set $V$ with $V \subseteq U$ such that $f(V) \subseteq G$. Then $V \subseteq f^{-1}(G)$ and so $V \subseteq \mathrm{Int}(f^{-1}(G))$. Thus $\emptyset \neq V = V \cap U \subseteq \mathrm{Int}(f^{-1}(G)) \cap U$, which implies that $\mathrm{Int}(f^{-1}(G)) \cap U \neq \emptyset$ for each semiopen $U$ containing $x$. Therefore $x \in \mathrm{Cl}_s(\mathrm{Int}(f^{-1}(G)))$. By Lemma 2.3 ($iii$), $x \in f^{-1}(G) \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(G))))$. Hence $f$ is $\alpha$-continuous. □

**Theorem 4.3.** *Let $X, Y$ be topological spaces. For a function $f : X \to Y$, the following are equivalent:*

*(1) $f$ is almost quasicontinuous;*

*(2) For each $x \in X$, each open set $G$ containing $f(x)$ and each open $U$ containing $x$, there exist a nonempty preopen set $V$ with $V \subseteq U$ such that $f(V) \subseteq G$;*

*(3) For each $x \in X$, each open set $G$ containing $f(x)$ and each open $U$ containing $x$, there exist a nonempty $\gamma$-open set $V$ with $V \subseteq U$ such that $f(V) \subseteq G$;*

*(4) For each $x \in X$, each open set $G$ containing $f(x)$ and each open $U$ containing $x$, there exist a nonempty $\beta$-open set $V$ with $V \subseteq U$ such that $f(V) \subseteq G$.*

*Proof. (1)⇒(2):* Let $G$ be an open set containing $f(x)$ and let $U$ be an open set containing $x$. By (1) and Lemma 2.3 ($ii$), $x \in f^{-1}(G) \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(G)))) = \mathrm{Cl}(\mathrm{Int}_p(f^{-1}(G)))$. This implies that $\mathrm{Int}_p(f^{-1}(G)) \cap O \neq \emptyset$ for each open $O$ containing $x$ and so $\mathrm{Int}_p(f^{-1}(G)) \cap U \neq \emptyset$. Set $V = \mathrm{Int}_p(f^{-1}(G)) \cap U$. By Lemma 2.4, $V$ is a nonempty preopen set with $V \subseteq U$ and

$$f(V) \subseteq f(\mathrm{Int}_p(f^{-1}(G)) \cap U) \subseteq f(f^{-1}(G)) \subseteq G.$$

This proves (2).

The implications "*(2)⇒(3)*" and "*(3)⇒(4)*" are follow from Remark 2.1.

*(4)⇒(1):* Let $x$ be any point in $X$ and let $G$ be an open set containing $f(x)$. By (4), for each open set $U$ containing $x$, there is a nonempty $\beta$-open set $V$ with $V \subseteq U$ such that $f(V) \subseteq G$. Then $V \subseteq f^{-1}(G)$ and so $V \subseteq \mathrm{Int}_\beta(f^{-1}(G))$. Thus $\emptyset \neq V = V \cap U \subseteq \mathrm{Int}_\beta(f^{-1}(G)) \cap U$, which implies that $\mathrm{Int}_\beta(f^{-1}(G)) \cap U \neq \emptyset$ for each open $U$ containing $x$. Therefore $x \in \mathrm{Cl}(\mathrm{Int}_\beta(f^{-1}(G)))$. By Lemma 2.3 ($ii$), $x \in f^{-1}(G) \subseteq \mathrm{Cl}(\mathrm{Int}_\beta(f^{-1}(G))) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(G))))$. This prove that $f$ is almost quasicontinuous. □

**Corollary 4.1.** *Let $X, Y$ be topological spaces. For a function $f : X \to Y$, the following are equivalent:*

*(1) $f$ is almost quasicontinuous;*

*(2) For each $x \in X$, each open set $G$ containing $f(x)$ and each open $U$ containing $x$, $f^{-1}(G) \cap U$ is swn-open;*

*(3) For each $x \in X$, each open set $G$ containing $f(x)$ and each $\alpha$-open $U$ containing $x$, there exists a nonempty open set $V$ with $V \subseteq U$ such that $V \subseteq \mathrm{Cl}(f^{-1}(G))$;*

*(4) For each $x \in X$, each open set $G$ containing $f(x)$ and each $\alpha$-open $U$ containing $x$, there exists a nonempty $\alpha$-open set $V$ with $V \subseteq U$ such that $V \subseteq \mathrm{Cl}(f^{-1}(G))$;*

*(5) For each $x \in X$, each open set $G$ containing $f(x)$ and each open $U$ containing $x$, there exists a nonempty $\alpha$-open set $V$ with $V \subseteq U$ such that $V \subseteq \mathrm{Cl}(f^{-1}(G))$;*

*(6) For each $x \in X$, each open set $G$ containing $f(x)$ and each open $U$ containing $x$, there exists a nonempty semiopen set $V$ with $V \subseteq U$ such that $V \subseteq \mathrm{Cl}(f^{-1}(G))$.*

*Proof.* Follows from Theorem 4.1 and Lemma 2.9. □

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