Divulgaciones Matemáticas Vol. 22, No. 1 (2021), pp. 31–39

The graph of a base power b, associated to a positive integer number

El grafo potencia de base b, asociado a un entero positivo

Daniel Brito (danieljosb@gmail.com)
Oscar Castro (oscarecastrop@gmail.com)
Lope Marín (lmata73@gmail.com)

Department of Mathematics, University of Oriente,
Cumaná, Bolivarian Republic of Venezuela.

Abstract

Many concepts of Number Theory were used in Graph Theory and several types of graphs have been introduced. We introduced the graph of a base power $b \in \mathbb{Z}^+ - \{1\}$, associated to a positive integer number $n \in \mathbb{Z}^+$, denoted for $GP_b(n)$, with set of vertices $V = \{x\}_{x=1}^n$ and with set of edges:

$$E = \{\{x, y\} \in 2^V : \exists r \in \mathbb{Z}^+ \cup \{0\}, \text{ such that } |y - x| = b^r\},$$

and we study some of its properties, in special for case $b = 2$.

Key words and phrases: Hamiltonian Cycle; Hamilton-connectivity; Pancyclicity.

Resumen

Muchos conceptos de la Teoría de Números han sido utilizados en la Teoría de Grafos y distintos tipos de grafos se han introducido. Introducimos el grafo de una potencia de base $b \in \mathbb{Z}^+ - \{1\}$, asociada a un entero $n \in \mathbb{Z}^+$, denotado por $GP_b(n)$, con conjunto de vértices $V = \{x\}_{x=1}^n$ y conjunto de lados

$$E = \{\{x, y\} \in 2^V : \exists r \in \mathbb{Z}^+ \cup \{0\}, \text{ tal que } |y - x| = b^r\},$$

y estudiamos algunas de sus propiedades, en especial para el caso $b = 2$.

Palabras y frases clave: Ciclo Hamiltoniano; Hamilton-conectividad; Panciclicidad.

1 Introduction

A *graph* G consist of a nonempty set $V(G)$ of elements represented for points, called *vertices* and a set $E(G)$ of elements represented for lines segments with ends an unique pair of vertices, this lines are called *edges* or *sides* and it is said that the pair of vertices are *adjacent*. A graph

Recibido 21/01/2021. Revisado 23/02/2021. Aceptado 30/05/2021.
MSC (2020): Primary 97N70, 05C45,; Secondary 05C38.
Autor de correspondencia: Oscar Castro

without *loops* (there is no an edge with equal ends vertices) and without *multiple edges* (there are no two or more edges with the same ends vertices) is called a *simple graph*. Two graph are *isomorphic* if both graphs have same properties and different graphics representation. Let $x \in V(G)$, the *degree* of x, denoted for $d_G(x)$, is the number of times that x is end of edges in G. Thus, $\delta(G)$ and $\Delta(G)$ denote, respectively, the *minimun degree* and *maximun degree* of G. A graph G is *complete* if every two distinct vertices in G are adjacent. A complete graph with n vertices is denoted by K^n. A *path* is a *subgraph* P of a graph G (graph with subsets of $V(G)$ and $E(G)$, respectively) formed by an alternating succession of adjacent vertices, furthermore, P has an initial vertex and final vertex, called *extreme*. A graph G is called *connected* if there is a path between any two distinct vertices in G. If there is no repetition of vertices in the path, it is said to be an *elemental* path. Let P be the path between the vertices x and y, also denoted by xPy or yPx indistinctly, but if we denoted xP^+y when P is traversed from x to y then we denoted xP^-y when P is traversed from y to x. A *Hamiltonian path* of a graph is an elemental path containing all the vertices of the graph. A *cycle* in a graph is an elemental path whose extreme vertices are the same. A *Hamiltonian cycle*, in a graph, is a cycle that visits each vertex of the graph. A *Hamiltonian graph* is a graph that contain a Hamiltonian cycle. A *Hamilton-connected graph* is a graph that contains a Hamiltonian path between each pair of vertices. A *pancyclic graph* is a graph that contains cycles of all the lengths, among 3 and n. In this paper only we consider simple graphs and we refer the reader to [3] for the definitions not given here.

The motivation of this paper is related with the use of the concepts of Number Theory and Graph Theory, to obtain other types of graphs as in [1] and [6]. We define a *graph associated to a positive integer number* n, denoted by $G(n)$, as a graph with set of vertices $V = \{x_k\}_{k=1}^n$, such that x_k is a succession in $\mathbb{C}^n$ and set of side $E = \{\{x_i, x_j\} \in 2^V / x_i \,\Phi\, x_j \vee i\Psi j\}$, with Φ, Ψ relations between x_i and x_j in V and between i, j in $\mathbb{Z}^+$, respectively, and 2^V the power set of V [5]. In particular, we introduce the *graph of a base power* $b \in \mathbb{Z}^+ - \{1\}$, associated to a positive integer number $n \in \mathbb{Z}^+$, denoted for $GP_b(n)$, with set of vertices $V = \{x\}_{x=1}^n$ and with set of edges:

$$E = \{\{x, y\} \in 2^V : \exists r \in \mathbb{Z}^+ \cup \{0\}, \text{ such that } |y - x| = b^r\},$$

and we characterized the degree for any vertex in $GP_b(n)$, the minimun degree and maximun grade of $GP_b(n)$ in function of n and b, moreover, we study the Hamilton-connectivity and the pancyclicity of $GP_2(n)$ and some other applications of the $GP_b(n)$ graph for case $n = 2$.

2 The graph of a power of a given base, associated to a positive integer

Let $d_{GP_b(n)}(x)$, $\delta(GP_b(n))$ and $\Delta(GP_b(n))$ be, respectively, the degree of x in $GP_b(n)$, the *minimun degree* and *maximun degree* of $GP_b(n)$.

Lemma 1. *For all $b \in \mathbb{Z}^+ - \{1\}$, for all $n \in \mathbb{Z}^+$ and for all $x \in V(GP_b(n))$, we have $d_{GP_b(n)}(x) = 0$, if $n = 1$ and for $n > 1$:*

$$d_{GP_b(n)}(x) = \begin{cases} \lfloor \log_b(x-1) \rfloor + \lfloor \log_b(n-x) \rfloor + 2, & \text{if } 1 < x < n \\ \lfloor \log_b(n-1) \rfloor + 1, & \text{if } x = 1,\, n \end{cases} .$$

Proof. Let $b \in \mathbb{Z}^+ - \{1\}$ and $n \in \mathbb{Z}^+$. If $n = 1$ then $GP_b(1) = K^1$, in consequence $d_{GP_b(1)}(x) = 0$, for $x \in V(GP_b(1))$. If $n > 1$, we have $q_n = \lfloor \log_b(n) \rfloor$, with $q_n = max\{i \in \mathbb{Z}^+ \cup \{0\} \,/\, b^i \leq n\}$

and, moreover, we consider two cases for $x \in V(GP_b(n))$, $x = 1, n$ or $x = h, l$, indistinctly (see Figure 1):

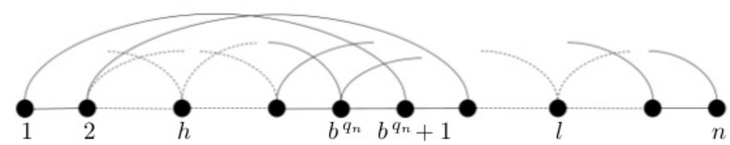

Figure 1: $GP_b(n)$, with $q_n = max\{i \in \mathbb{Z}^+ \cup \{0\} \,/\, b^i \leq n\}$.

Case 1. If $x = 1$, then by definition of $GP_b(n)$, x is adjacent

$$\underbrace{2, b+1, b^2+1, \cdots, b^r+1}_{r+1}$$

or if $x = n$, then by definition of $GP_b(n)$, x is adjacent to

$$\underbrace{n-1, n-b, n-b^2, \cdots, n-b^r}_{r+1}.$$

Thus, as $b^r + 1 \leq n$, we have $d_{GP_b(n)}(x) = r + 1$, if

$$r = \begin{cases} q_n, & \text{if } n > b^{q_n} \\ q_n - 1, & \text{if } n = b^{q_n} \end{cases}.$$

But for each $b, n \in \mathbb{Z}^+ - \{1\}$,

$$\lfloor \log_b(n-1) \rfloor = \begin{cases} q_n, & \text{if } n > b^{q_n} \\ q_n - 1, & \text{if } n = b^{q_n} \end{cases}. \tag{1}$$

Therefore, $d_{GP_b(n)}(x) = \lfloor \log_b(n-1) \rfloor + 1$, for $x = 1, n$.

Case 2. If $x = h$ or $x = l$, indistinctly, then $1 < x < n$ (see Figure 1) and by definition of $GP_b(n)$, x is adjacent to

$$\underbrace{x-b^r, \cdots, x-b^2, x-b, x-1}_{r+1}$$

and to

$$\underbrace{1+x, b+x, b^2+x, \cdots, b^s+x}_{s+1}.$$

Thus, as $1 \leq x - b^r$ and $b^s + x \leq n$, we have $d_{GP_b(n)}(x) = r + s + 2$, if

$$r = \begin{cases} q_x, & \text{if } x > b^{q_x} \\ q_x - 1, & \text{if } x = b^{q_x} \end{cases}$$

and $s = q_{n-x}$. But by equation 1, it follows that $q_x = max\{i \in \mathbb{Z}^+ \cup \{0\} \,/\, b^i \leq x\}$, furthermore, $q_{n-x} = max\{i \in \mathbb{Z}^+ \cup \{0\} \,/\, b^i \leq n - x\}$. Therefore, we obtain that $d_{GP_b(n)}(x) = \lfloor \log_b(x-1) \rfloor + \lfloor \log_b(n-x) \rfloor + 2$. □

Theorem 1. *For all $b \in \mathbb{Z}^+ - \{1\}$, for all $n \in \mathbb{Z}^+$ and for all $x \in V(GP_b(n))$, we have:*

1. $d_{GP_b(n)}(x) = d_{GP_b(n)}(n - x + 1)$.
2. $\delta(GP_b(n)) = d_{GP_b(n)}(1) = d_{GP_b(n)}(n)$.
3. $\Delta(GP_c(n)) \leq \lfloor 2[\log_b(\frac{n-1}{2}) + 1] \rfloor$, *if* $n \geq 3$.

Proof. (Numeral 1). Follows from Lemma 1.

(Numeral 2). Let $b \in \mathbb{Z}^+ - \{1\}$. If $n = 1$ then $GP_b(1) = K^1$, in consequence $d_{GP_b(1)}(x) = 0$, for $x \in V(GP_b(1))$. Thus, $\delta(GP_b(1)) = \Delta(GP_b(1)) = 0$. Furthermore, if $n = 2$ then $GP_b(2) = K^2$, therefore $d_{GP_b(2)}(x) = 1$, for all $x \in V(K^2)$, in consequence $\delta(GP_b(2)) = \Delta(GP_b(2)) = 1$. This is, $\delta(GP_b(1)) = d_{GP_b(1)}(1) = 0$ and $\delta(GP_b(2)) = d_{GP_b(2)}(1) = d_{GP_b(2)}(2) = 1$, for all $b \in \mathbb{Z}^+ - \{1\}$.

If $n \geq 3$, by the proof of Numeral 1, is sufficient that $1 < x \leq \lfloor \frac{n+1}{2} \rfloor$ for prove that $d_{GP_b(n)}(1) = d_{GP_b(n)}(n) \leq d_{GP_b(n)}(x)$, for all $x \in V(GP_b(n))$. We consider two cases for $x \in V(GP_b(n))$, $x = 2$ or $3 \leq x \leq \lfloor \frac{n+1}{2} \rfloor$:

Case 1. If $x = 2$, by Lemma 1 and the equation 1, we have $d_{GP_b(n)}(2) = \lfloor \log_b(n-2) \rfloor + 2$ and $q_n - 1 \leq \lfloor \log_b(n-2) \rfloor \leq \lfloor \log_b(n-1) \rfloor \leq q_n$. Therefore $\lfloor \log_b(n-2) \rfloor + 2 \geq \lfloor \log_b(n-1) \rfloor + 1 = d_{GP_b(n)}(1) = d_{GP_b(n)}(n)$.

Case 2. If $3 \leq x \leq \lfloor \frac{n+1}{2} \rfloor$ and n is even, then $3 \leq x \leq \frac{n}{2} = \lfloor \frac{n+1}{2} \rfloor$, so that $3 \leq x \leq n - x$ and $n - 1 \leq (n - x)(x - 1)$ ($w, z \in \mathbb{Z}$ and $2 \leq w \leq z \Rightarrow w + z \leq wz$), in consequence $\log_b(n-1) \leq \log_b(n-x) + \log_b(x-1)$.

If $3 \leq x \leq \lfloor \frac{n+1}{2} \rfloor$ and n is odd, then $3 \leq x \leq \frac{n}{2} < \lfloor \frac{n+1}{2} \rfloor = \frac{n+1}{2}$, in consequence we need to prove, only that $x = \frac{n+1}{2}$ implies $\log_b(n-1) \leq \log_b(n-x) + \log_b(x-1)$. Indeed, if $n = 2k+1$, with $k \in \mathbb{Z}^+ - \{1\}$ ($3 \leq x \leq \frac{n+1}{2}$), then $x = \frac{n+1}{2} = k+1$ and as $2k \leq k^2, \forall k \in \mathbb{Z}^+ - \{1\}$, similarity, we obtain that $n - 1 \leq (n - x)(x - 1)$. Therefore, $\log_b(n-1) \leq \log_b(n-x) + \log_b(x-1)$.

Likewise, as $\lfloor w \rfloor + \lfloor z \rfloor \leq \lfloor w + z \rfloor \Leftrightarrow \lfloor w + z \rfloor + 1 \leq \lfloor w \rfloor + \lfloor z \rfloor + 2, \forall w, z \in \mathbb{R}$ then we have $\lfloor \log_b(n-1) \rfloor + 1 \leq \lfloor \log_b(x-1) \rfloor + \lfloor \log_b(n-x) \rfloor + 2$. This is, $d_{GP_b(n)}(1) = d_{GP_b(n)}(n) \leq d_{GP_b(n)}(x)$, if $3 \leq x \leq \lfloor \frac{n+1}{2} \rfloor$ (see Lemma 1).

(Numeral 3). If $n = 3$ we consider that $\Delta(K^3) = 2 = \lfloor 2[\log_b(\frac{3-1}{2}) + 1] \rfloor$, for all $b \in \mathbb{Z}^+ - \{1\}$.
If $n = 4$ then, by definition of $GP_b(n)$,

$$\Delta(GP_b(4)) = \left\lfloor 2 \left[\log_b \left(\frac{4-1}{2} \right) + 1 \right] \right\rfloor = \begin{cases} 3, & \text{if } b = 2 \\ 2, & \text{if } b > 2 \end{cases} .$$

If $n \geq 5$, we consider that the maximum of the function $f(x) = \log_b(x-1) + \log_b(n-x) + 2$, in the interval $[2, n-1]$, is $2\log_b(\frac{n-1}{2}) + 2$ for $x = \frac{n+1}{2}$, ($f(x)$ is continuous and differentiable function in $]2, n-1[$ and $[2, n-1]$, respectively), whereby $f(x)$ reach the maximum valor (Weierstrass's Extreme Valor Theorem and Critical Value of f [4]. Furthermore, we consider the proof of Numeral 1 and the Lemma 1. Thus, we have $\Delta(GP_b(n)) \leq \lfloor 2[\log_b(\frac{n-1}{2}) + 1] \rfloor$. □

We consider other interesting properties. For all $b \in \mathbb{Z}^+ - \{1\}$ and for all $n \in \mathbb{Z}^+$, $GP_b(n) \subseteq GP_b(n+1)$ ($GP_b(n)$ is subgraph of $GP_b(n+1)$). Furthermore, $GP_b(n)$ contain a Hamiltonian path, denoted by HP, such that for $n > 1$, $E(HP) = \{\{x, x+1\}\}_{x=1}^{n-1}$. However, some graphs of base power b, associated to a positive integer number n, no contain a Hamiltonian cycle, for example $GP_3(9)$ contain a Hamiltonian path HP: 1,2,3,4,5,6,7,8,9 and no contain a Hamiltonian cycle (see Figure 2).

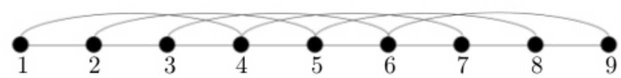

Figure 2: $GP_3(9)$.

Otherwise, if $n = b^s + b^r$, with $n \geq 3$, $b \in \mathbb{Z}^+ - \{1\}$, $s, r \in \mathbb{Z}^+ \cup \{0\}$ and, without loss of generality, $q_n = r$ (see equation 1) then $HC : n, HP^-, b^r+1, 1, HP^+, b^r, n$ is a Hamiltonian cycle in $GP_b(n)$. For example $GP_3(12)$ contain a Hamiltonian cycle $HC : 12, 11, 10, 1, 2, 3, 4, 5, 6, 7, 8, 9, 12$, because $n = 12 = 3 + 3^2$ (see Figure 3).

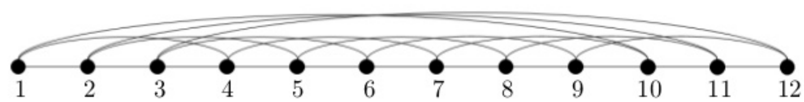

Figure 3: $GP_3(12)$.

In consequence, for any $B = b$ in $\mathbb{Z}^+$, if $A = r = s = 2$ or if $A = b+1$, with $r = b+2$ and $s = b+1$, then $GP_B(AB^A)$ is Hamiltonian. This is, the $GP_b(2b^2)$ and $GP_b((b+1)b^{b+1})$ are Hamiltonian. Thus, there exists an infinite subsuccession, in the ABA *numbers* sequence (have the form ab^a, $\forall a \in \mathbb{Z}^+$), associated to a Hamiltonian graph of a base power B to a positive integer number ABA. The ABA numbers, sequence A171607 in the OEIS (The On-Line Encyclopedia of Integer Sequences) [7], is a generalization of the Cullen and Woodall numbers. The *Cullen numbers* are given by the expression $a2^a + 1$, (sequence A002064 in the OEIS, [8]) and *Woodall numbers* by $a2^a - 1$, $\forall a \in \mathbb{Z}^+$ (sequence A003261, [9]).

Furthermore, for $n \in \mathbb{Z}^+ - \{1\}$ fixed, if $b \to \infty$ then $GP_b(n) \to HP^{n-1}$, this is, if $b > n$ implies that the graph $GP_b(n)$ is the Hamiltonian path HP with length $n-1$ (Lemma 1 and the proof of Numeral 1 in the Theorem 1). For example, for n = 4, see Figure 4.

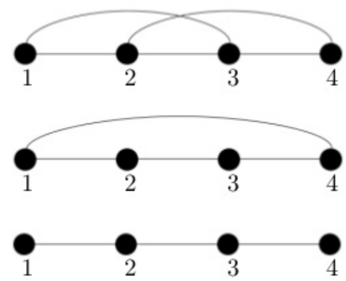

Figure 4: $b \to \infty \Rightarrow GP_b(4) \to HP^3$.

Also, thanks to the Lemma 1, we can obtain the degree sequence of any $GP_b(n)$. A non-decreasing sequence $q_l, q_2, \ldots, q_n$ of non-negative integers is the *degree sequence* or *graphic sequence*, if only if there is a graph G with n vertices $x_l, x_2, \ldots, x_n$, such that the $d_G(x_i) = q_i$ for $i = 1, 2, \ldots, n$ [2].

3 The graph $GP_2(n)$

In this section, we studied the graphs of base power 2, associated to a positive integer number n, when are pancyclic, Hamiltonian and Hamilton-connected.

Theorem 2. *For all integer $n \geq 3$, $GP_2(n)$ is pancyclic.*

Proof. For $n = 3$, $GP_2(3) = K^3$, which is pancyclic (see Figure 5, below).

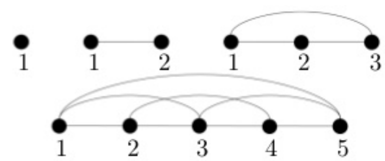

Figure 5: $GP_2(n)$, with $n = 1, 2, 3, 5$.

Let $n \geq 4$. By definition of $GP_2(n)$, there is a Hamiltonian path, HP in $GP_2(n)$, with initial vertex 1 and final vertex n, its vertices set contain a consecutive positives integer succession. Likewise, $GP_2(n)$ have sides $\{x_i, x_{i+1}\}$ for $i = 1, 2, 3, \ldots, p_k$ and $\{y_j, y_{j+1}\}$ for $j = 1, 2, 3, \ldots, q_k$, with $x_i = 2i - 1$, $y_j = 2j$, $p_k = \left\lfloor \frac{k}{2} \right\rfloor - par(k)$, $q_k = \left\lfloor \frac{k}{2} \right\rfloor - 1$, for $k = 4, 5, 6, \ldots, n$, and $par(h) = \frac{1+(-1)^h}{2}$ (*parity* of $h \in \mathbb{Z}$). Therefore, we consider in $GP_2(n)$ the $n - 3$ cycles:

$$C^k : x_1, x_2, \cdots, x_{p_k+1}, y_{q_k+1}, \cdots, y_2, y_1, x_1,$$

furthermore, we observe that $V(C^k) = \{x_i\}_{i=1}^{p_k+1} \cup \{y_j\}_{j=1}^{q_k+1}$, $|V(C^k)| = k$ and as $K^3 = GP_2(3) \subseteq GP_2(n)$, for all integer $n \geq 3$, $GP_2(n)$ also contain the cycle $C^3 : 1, 3, 2, 1$, thus, we obtain $GP_2(n)$ is pancyclic. □

We observe that for $k = n$, in the proof of Theorem 2, $GP_2(n)$ contains a Hamiltonian cycle, therefore, for all $n \geq 3$, $GP_2(n)$ is a Hamiltonian graph. For example, $GP_2(4)$ and $GP_2(5)$ are isomorphic, respectively, to the graphs in Figure 6.

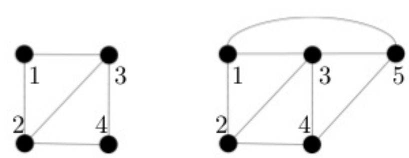

Figure 6: The isomorphic of $GP_2(n)$, with $n = 4, 5$.

We observe the cycles in $GP_2(4)$:

$$C^3\text{: 1,3,2,1} \quad C^4\text{: 1,3,4,2,1.}$$

And observe the cycles in $GP_2(5)$:

Theorem 3. *For all $n \in \mathbb{Z}^+ - \{4\}$, $GP_2(n)$ is Hamilton-connected.*

$$C^3\text{: } 1,3,2,1 \quad C^4\text{: } 1,3,4,2,1 \quad C^5\text{: } 1,3,5,4,2,1.$$

Proof. For $n = 1,2,3$, follows from the definition of $GP_2(n)$, moreover, $GP_2(1)$, $GP_2(2)$ and $GP_2(3)$ are isomorphic to the complete graph K^1, K^2 and K^3, respectively (see Figure 5, further behind). Thus, for $n > 4$ we apply induction over n.

If $n = 5$, we consider the elemental paths in $GP_2(5)$ (observing Figure 6):

HP^4: 1,2,3,4,5	$2P5$: 2,1,3,4,5	$3P5$: 3,1,2,4,5	$4P5$: 4,3,2,1,5
$1P4$: 1,2,3,5,4	$2P4$: 2,1,3,5,4	$3P4$: 3,2,1,5,4	
$1P3$: 1,2,4,5,3	$2P3$: 2,1,5,4,3		
$1P2$: 1,3,5,4,2			

Then, $GP_2(5)$ is Hamilton-connected.

If $n = 6$ (see Figure 7), we consider four cases, based in the construction of the Hamiltonian paths in $GP_2(5)$:

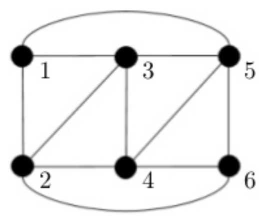

Figure 7: $GP_2(6)$.

Case 1. The elemental paths in $GP_2(6)$, expanding the Hamiltonian paths in $GP_2(5)$ define previously:

$1P5$: 1,2,3,4,**6**,5	$2P5$: 2,1,3,4,**6**,5	$3P5$: 3,1,2,4,**6**,5
$1P4$: 1,2,3,5,**6**,4	$2P4$: 2,1,3,5,**6**,4	$3P4$: 3,2,1,5,**6**,4
$1P3$: 1,2,4,**6**,5,3	$2P3$: 2,1,5,**6**,4,3	
$1P2$: 1,3,5,**6**,4,2		

Case 2. For the Hamiltonian paths HP^5, $2P6$, $3P6$, $4P6$ and $5P6$ in $GP_2(6)$, we extend the Hamiltonian paths in $GP_2(5)$ to:

HP^5: 1,2,3,4,5,**6** $2P6$: 2,1,3,4,5,**6** $3P6$: 3,1,2,4,5,**6** $4P6$: 4,3,2,1,5,**6**

Case 3. For the Hamiltonian path $4P5$ in $GP_2(6)$, we extend the Hamiltonian path $4P2$ in $GP_2(4)$ (see Figure 6), therefore $4P5 : 4,3,1,2,\mathbf{6},\mathbf{5}$.

Case 4. For the Hamiltonian path $5P6$ in $GP_2(6)$, we extend the Hamiltonian path $2P5$ in $GP_2(5)$ (see Figure 6), therefore $5P6 : 5,4,3,1,2,\mathbf{6}$.

Thus, $GP_2(6)$ is Hamilton-connected.

Successively, by construction, from the Hamiltonian paths in $GP_2(5)$, suppose that theorem is true for $6 \leq n = h$ (inductive hypothesis: $GP_2(h)$ is Hamilton-connected), we will demonstrate that $GP_2(h+1)$ is Hamilton-connected:

If we consider $n = h+1$, all elemental paths in $GP_2(h+1)$, with length h, contain h vertices of any elemental paths in $GP_2(h)$ and contain $h-1$ vertices of any elemental paths in $GP_2(h-1)$ ($GP_2(h-1) \subseteq GP_2(h) \subseteq GP_2(h+1)$). Therefore, for $x, y \in V(GP_2(h+1))$ different, without loss of generality, suppose $x < y$, we chosen the vw-elemental path in $GP_2(h)$ (or in $GP_2(h-1)$), and we consider four cases, for found the xy-elemental path, with length h, in $GP_2(h+1)$:

Case 1. If, simultaneously, $v \neq h-1$, $w \neq h$ and $x, y \neq h+1$, then by construction, from the Hamiltonian paths in $GP_2(5)$ until $GP_2(h)$, in $GP_2(h)$, we obtain the elemental paths with length $h-1$:

$$vPw : v, \cdots, h-1, h, \cdots, w$$

or

$$vPw : v, \cdots, h, h-1, \cdots, w$$

which are expanded, respectively, to the elemental paths in $GP_2(h+1)$, with length h:

$$xPy : v, \cdots, h-1, \mathbf{h+1}, h, \cdots, w$$

or

$$xPy : v, \cdots, h, \mathbf{h+1}, h-1, \cdots, w.$$

Case 2. Let $v = x$, if $w = h$ then $y = h+1$, so that, in $GP_2(h)$, we obtain the elemental path, with length $h-1$ (inductive hypothesis):

$$vPw : v, \cdots, w.$$

which is extended to, the path in $GP_2(h+1)$, with length h:

$$xPy : v, \cdots, w, \mathbf{h+1}.$$

Case 3. If $x = h-1$ then $y = h$. Thus, we chosen the vw-Hamiltonian path in $GP_2(h-1)$ (for $5 \leq h-1 < h$, we consider the inductive hypothesis), with $v = h-1$ and $w = h+1-2^r$ for $r > 1$, and we obtain the elemental path, with length h:

$$xPy : v, \cdots, w, \mathbf{h+1}, \mathbf{h}.$$

Case 4. If $x = h$ and $y = h+1$, we chosen the vw- Hamiltonian path in $GP_2(h)$, with $v = h$ and $w = h+1-2^r$ for $r > 1$ (inductive hypothesis), we obtain the elemental path, with length h:

$$xPy : v, \cdots, w, \mathbf{h+1}.$$

Finally, if $n = 4$, we observe in the Figure 6 above, that in $GP_2(4)$ does not exists a Hamiltonian path $2P3$, in consequence $GP_2(n)$ is Hamilton-connected for all $n \in \mathbb{Z}^+ - \{4\}$. □

We observe that the graph $GP_2(n)$ has interesting properties obtained by construction, without the need for many conditions.

Given Theorem 2 and Theorem 3, for any integer $n \geq 16$, $GP_2(n)$ are examples of an infinity of Hamiltonian graphs, such that $\frac{k}{k+1}n \geq \frac{n}{2} > \Delta(GP_2(n))$, with $k \in \mathbb{Z}^+$ or $n > 2\Delta(GP_2(n)) \geq d_{GP_2(n)}(x) + d_{GP_2(n)}(y)$ for any $x, y \in V(GP_2(n))$. In all cases, we consider Lemma 1, Theorem 1 and $\sqrt[4]{2}(n-1) > n$, for all integer $n \geq 7$ and $2^{\frac{n}{2}} > (n-1)^2$, for all integer $n \geq 16$. In consequence, the hypothesis of Seymour's Conjecture, of Dirac's Theorem and of Ore's Theorem (see [3]) are no necessary for the graph $GP_2(n)$ with $n \geq 16$. Furthermore, by Lemma 1, Theorem 2 and the inequalities shows above, we obtain that the sequence degree of $GP_2(n)$, for all integer $n \geq 16$, no satisfies the hypothesis of Chvátal's Theorem [2]. In consequence, $\{GP_2(n)\}_{n=16}^{\infty}$ is a succession of Hamiltonian graphs whose degree sequence is majorized by a graphic sequence which is not forcibly Hamiltonian. A sequence $q_1, q_2, \ldots, q_k$ *majorizes* a sequence $d_1, d_2, \ldots, d_k$ if and only if $q_i \geq d_i$, for all $i \leq k$, with $k \in \mathbb{Z}^+$ and a graphic sequence is *forcibly Hamiltonian*, if and only if every graph with this degree sequence is Hamiltonian [2].

Finally, thanks to $GP_2(n)$, we show that any sequence of consecutive positive integers, $\{n\}_{n=1}^{m}$, is associated, simultaneously, to a nontrivial pancyclic (Hamiltonian) and Hamilton-connected graph.

References

[1] Abawajy, J., Kelarev, A. and Chowdhury, M., *Power Graphs: A Survey*, Electronic Journal of Graph Theory and Applications, **1**(2)(2013), 125-147.

[2] Chvátal, V., *On Hamilton's Ideal*, Journal of Combinatorial Theory, **12**(B)(1972), 163–168.

[3] Diestel, R., *Graph Theory* (2th ed.), Springer, New York, 2000.

[4] Leithold, L., *El Cálculo* (7th ed.), Oxford University Press, México, 1998.

[5] Lipschutz, S., *Teoría de Conjuntos y Temas Afines*, McGraw-Hill/Interamericana de México S. A. de C.V., México, 1994.

[6] Mukherjee, H., *Hamiltonicity of the Power Graph of Abelian Groups*, in :`https://arxiv.org/pdf/1505.00584`(2017), consulted 05/07/2017, 01:25 a.m.

[7] Munafo, R., *Expressible as A^*B^A in a nontrivial way*, in: `https://oeis.org/A171607`(2009) consulted 20/11/2019, 02:14 a.m.

[8] Sloane, N., *Cullen numbers: $n * 2^n + 1$*, in :`https://oeis.org/A002064`(2012), consulted 20/11/2019, 00:14 a.m.

[9] Sloane, N., *Woodall (or Riesel) numbers: $n * 2^n - 1$*, in: `https://oeis.org/A003261`(2012), consulted 20/11/2019, 01:05 a.m.