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Existence of positive solutions of BVPs for coupled impulsive differential equations on whole line with mixed boundary conditions

Existencia de soluciones positivas de problemas con valores en la frontera para ecuaciones diferenciales con impulso acopladas sobre toda la recta con condiciones de frontera mixta.

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Abstract

This paper is concerned with boundary value problems of impulsive differential systems on whole lines with nonlinear differential operators. By constructing a weighted Banach space and defining a nonlinear operator, using the Schauder's fixed point theorem and Schaefer's fixed point theorem, sufficient conditions to guarantee the existence of at least one positive solution are established. An example is given to illustrate the main results.

Key words and phrases: Impulsive differential system on whole line, boundary value problem, increasing odd homeomorphisms, sub-Carathéodory function, discrete Carathéodory function, fixed point theorem.

Resumen

Este artículo está interesado en problemas con valores en la frontera para sistemas diferenciales con impulso sobre la linea recta con operadores diferenciales no lineales. Construyendo un espacio de Banach ponderado y definiendo un operador no lineal, usando el teorema del punto fijo de Schauder y el teorema del punto fijo de Schaefer, se establecen condiciones suficientes para garantizar la existencia de al menos una solución positiva. Se da un ejemplo para ilustrar los principales resultados.

Palabras y frases clave: Sistemas diferenciales con impulso, problemas con valores en la frontera, homeomorfismos impares crecientes, función sub-Catathéodory, función Carathéodory discreta, teorema del punto fijo.

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1 Introduction

Boundary value problems for second order ordinary differential equations (ODEs) were initiated by Il'in and Moiseev [25] and studied by many authors, see the text books [20, 27], the papers [14, 16, 33] and the references therein.

The asymptotic theory of ordinary differential equations is an area in which there is great activity among a large number of investigators since it has many applications in real world applications [21, 22, 23, 24, 9]. In this theory, it is of great interest to investigate, in particular, the existence of solutions with prescribed asymptotic behavior, which are global in the sense that they are solutions on the whole line. The existence of global solutions with prescribed asymptotic behavior is usually formulated as the existence of solutions of boundary value problems on the whole line.

In [13], the existence and multiplicity of nonnegative solutions for the following integral boundary value problem on the whole line were studied:

$$\begin{aligned} &(p(t)x'(t))' + \lambda q(t) f(t, x(t), x'(t)) = 0, \quad t \in \mathbb{R}, \\ &a_1 \lim_{t\to-\infty} x(t) - b_1 \lim_{t\to-\infty} p(t)x'(t) = \int_{-\infty}^{\infty} g_1(s, x(s), x'(s))\psi(s)ds, \\ &a_2 \lim_{t\to+\infty} x(t) + b_2 \lim_{t\to+\infty} p(t)x'(t) = \int_{-\infty}^{\infty} g_2(s, x(s), x'(s))\psi(s)ds, \end{aligned} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $f, g_1, g_2 \in C(\mathbb{R} \times [0,\infty) \times \mathbb{R}, [0,+\infty))$, $q, \psi \in C(\mathbb{R}, (0,+\infty))$ and $p \in C(\mathbb{R}, (0,+\infty)) \cap C^1(\mathbb{R})$. Here, the values of $\int_{-\infty}^{+\infty} g_i(s, x(s), x'(s))ds$ $(i = 1, 2)$, $\int_{-\infty}^{+\infty} \frac{ds}{p(s)}$ and $\sup_{s\in\mathbb{R}} \psi(s)$ are finite and $a_1 + a_2 > 0$, $b_i > 0$ $(i = 1, 2)$ satisfying $D = a_2 b_1 + a_1 b_2 + a_1 a_2 \int_{-\infty}^{+\infty} \frac{ds}{p(s)} > 0$.

In recent years, many authors have studied the existence of positive radial solutions for elliptic systems in annular/exterior domains, which is equivalent to that of positive solutions for the corresponding systems of ordinary differential equations (see [11, 30, 19, 18, 10, 29] and the references therein). The usual method used is the fixed point theorems of cone expansion/compression type, the upper and lower solutions method and the fixed point index theory in cones.

In [28, 12], the following system and its special case were discussed:

$$\begin{aligned} &[\phi_p(u'(t))]' + \lambda h_1(t) f(t, u(t), v(t)) = 0, \quad t \in (0,1), \\ &[\phi_p(v'(t))]' + \mu h_2(t) g(t, u(t), v(t)) = 0, \quad t \in (0,1), \\ &u(0) = a \geq 0, v(0) = b \geq 0, u(1) = v(1) = 0, \end{aligned} \tag{1.2}$$

where $\phi_p(x) = |x|^{p-2}x, p > 1$, λ, μ are nonnegative real parameters, $h_i \in C((0,1), (0,\infty)), i = 1, 2$, $f, g \in C([0,\infty) \times [0,\infty), [0,\infty))$, h_i may be singular at $t = 0$ and $f(0,0) = g(0,0) = 0$ and $f(u,v) > 0, g(u,v) > 0$ for all $(u,v) > (0,0)$. The existence, nonexistence and multiplicity of positive solutions for (1.1) were obtained by using the upper and lower solution method and the fixed point index theorem.

Theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, for example, phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction of the basic theory of impulsive differential equation, we refer the reader to [26].

In [32], Liu studied the existence of solutions of the following boundary value problem for second order impulsive differential system on the whole line with Dirichlet boundary conditions

$$\begin{aligned} &[\rho(t)\Phi_p(x'(t))]' + f(t, x(t), y(t)) = 0, \quad a.e.\ t \in \mathbb{R}, \\ &[\varrho(t)\Phi_q(y'(t))]' + g(t, x(t), y(t)) = 0, \quad a.e.\ t \in \mathbb{R} \\ &\lim_{t\to+\infty} x(t) = 0, \ \lim_{t\to-\infty} x(t) = 0, \ \lim_{t\to+\infty} y(t) = 0, \ \lim_{t\to-\infty} y(t) = 0, \\ &\Delta x(t_k) = I_k(t_k, x(t_k), y(t_k)), \ \ \Delta y(t_k) = J_k(t_k, x(t_k), y(t_k)), k \in \mathbb{Z}, \end{aligned} \tag{1.3}$$

where $\rho, \varrho \in C^0(\mathbb{R}, [0, \infty))$, $\rho(t), \varrho(t) > 0$ for all $t \in \mathbb{R}$ with

$$\int_{-\infty}^{+\infty} \frac{ds}{\rho(s)} < +\infty, \ \int_{-\infty}^{+\infty} \frac{ds}{\varrho(s)} < +\infty, \tag{1.4}$$

$\Phi_p(x) = |x|^{p-2}x$ and $\Phi_q(x) = |x|^{q-2}x$ are one-dimensional p-Laplacian, f, g defined on $\mathbb{R}^3$ are Carathéodory functions, $\cdots < t_k < t_{k+1} < t_{k+2} < \cdots$ with

$$\lim_{k\to-\infty} t_k = -\infty, \ \lim_{k\to+\infty} t_k = +\infty,$$

$\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $\Delta y(t_k) = y(t_k^+) - y(t_k^-)(k \in \mathbb{Z})$, $\{I_k\}, \{J_k\}$ with $I_k, J_k : \mathbb{R}^3 \to \mathbb{R}(k \in \mathbb{Z})$ are discrete Carathéodory sequences.

This paper is a continuation of [32]. We consider the existence of positive solutions of the following boundary value problem for second order differential system on the whole line with mixed boundary conditions with impulse effects:

$$
\begin{aligned}
&[\Phi(\rho(t)x'(t))]' + p(t)f(t,y(t),y'(t)) = 0, \quad a.e.\ t\in\mathbb{R},\\
&[\Psi(\varrho(t)y'(t))]' + q(t)g(t,x(t),x'(t)) = 0, \quad a.e.\ t\in\mathbb{R}\\
&\lim_{t\to-\infty} x(t)=0,\ \ \lim_{t\to+\infty}\rho(t)x'(t)=0,\\
&\lim_{t\to-\infty} y(t)=0,\ \ \lim_{t\to+\infty}\varrho(t)y'(t)=0,\\
&\Delta x(t_s) = A_{0,s}I_0(t_s,y(t_s),y'(t_s)),\ \Delta\Phi(\rho(t_s)x'(t_s)) = A_{1,s}I_1(t_s,y(t_s),y'(t_s)), s\in\mathbb{Z},\\
&\Delta y(t_s) = B_{0,s}J_0(t_s,x(t_s),x'(t_s)),\ \Delta\Psi(\varrho(t_s)y'(t_s)) = B_{1,s}J_1(t_s,x(t_s),x'(t_s)), s\in\mathbb{Z},
\end{aligned}
\tag{1.5}
$$

where

(a) $\rho,\varrho\in C^0(\mathbb{R},[0,+\infty))$ with

$$
\begin{aligned}
&\textstyle\int_{-\infty}^{0}\frac{ds}{\rho(s)}ds<+\infty,\ \int_0^{+\infty}\frac{ds}{\rho(s)}ds=+\infty,\\
&\textstyle\int_{-\infty}^{0}\frac{ds}{\varrho(s)}ds<+\infty,\ \int_0^{+\infty}\frac{ds}{\varrho(s)}ds=+\infty,
\end{aligned}
\tag{1.6}
$$

(b) $p,q\in C^0(\mathbb{R},(0,\infty))$ with

$$
\begin{aligned}
&\textstyle\int_0^{+\infty}p(s)ds<+\infty,\ \int_{-\infty}^{0}p(s)ds=+\infty,\\
&\textstyle\int_0^{+\infty}q(s)ds<+\infty,\ \int_{-\infty}^{0}q(s)ds=+\infty,
\end{aligned}
\tag{1.7}
$$

(c) $\Phi(x)=\Phi_{p_1}(x)=|x|^{p_1-2}x$ and $\Psi(x)=\Phi_{p_2}(x)=|x|^{p_2-2}x$ are one-dimensional p-Laplacians, their inverse operator are defined by Φ^{-1} and Ψ^{-1}, respectively, with $\Phi^{-1}(x)=|x|^{q_1-2}x$ and $\Psi^{-1}(x)=|x|^{q_2-2}x$, $\frac{1}{p_i}+\frac{1}{q_i}=1$,

(d) f defined on $\mathbb{R}^3$ strongly ϱ-Carathéodory function (see Definition 2.1), g defined on $\mathbb{R}^3$ strongly ρ-Carathéodory function (see Definition 2.2), f,g are nonnegative functions, and $[p(t)f(t,0,0)]^2+[q(t)g(t,0,0)]^2>0$ on each subinterval of $\mathbb{R}$,

(e) $\cdots<t_k<t_{k+1}<t_{k+2}<\cdots$ with $\lim_{k\to-\infty}t_k=-\infty$, $\lim_{k\to+\infty}t_k=+\infty$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$ and $\Delta y(t_k)=y(t_k^+)-y(t_k^-)(k\in\mathbb{Z})$, $\Delta x'(t_k)=x'(t_k^+)-x'(t_k^-)$ and $\Delta y'(t_k)=y'(t_k^+)-y'(t_k^-)(k\in\mathbb{Z})$,

(f) $I_0,I_1:\{t_s:s\in\mathbb{Z}\}\times\mathbb{R}^2\to\mathbb{R}$ are discrete $\varrho-$Carathéodory functions (see Definition 2.3), $J_0,J_1:\{t_s:s\in\mathbb{Z}\}\times\mathbb{R}^2\to\mathbb{R}$ is a discrete $\rho-$Carathéodory function (see Definition 2.4), I_0,J_0 are nonnegative functions, I_1,J_1 are non-positive functions,

(g) $A_{0,s}, A_{1,s}, B_{0,s}, B_{1,s} \geq 0$ satisfy

$$\sum_{j=-\infty}^{s} A_{0,j} < +\infty,\ \sum_{j=-\infty}^{s} B_{0,j} < +\infty,\ s \in \mathbb{Z},$$

$$\lim_{s\to+\infty} \frac{A_{0,s}}{\int_{t_{s-1}}^{t_s} \frac{ds}{\rho(s)}} = \lim_{s\to+\infty} \frac{B_{0,s}}{\int_{t_{s-1}}^{t_s} \frac{ds}{\varrho(s)}} = 0,$$

$$\sum_{j=s}^{+\infty} A_{1,j} < +\infty,\ \sum_{j=s}^{+\infty} B_{1,j} < +\infty,\ s \in \mathbb{Z},$$

$$\lim_{s\to-\infty} \frac{A_{1,s}}{\int_{t_s}^{t_{s+1}} p(s)ds} = \lim_{s\to-\infty} \frac{B_{1,s}}{\int_{t_s}^{t_{s+1}} q(s)ds} = 0.$$

The purpose of this paper is to establish sufficient conditions for the existence of at least one positive solution of BVP(1.5). The technical tool used in this paper is the well known Schauder fixed point theorem. For applying this theorem, the most crucial things are to construct a nonlinear operator and to prove the compactness property of the nonlinear operator. Since the problem is considered on whole line, we need to show the equi-continuous properties of the image of a bounded set on each sub intervals (there are infinitely many sub intervals), the equi-convergence as $t \to t_i (i = 0, \pm 1, \pm 2, \dots)$ and the equi-convergence as $t \to -\infty$ and as $t \to +\infty$. One sees from (1.6) that both $\frac{1}{\rho}$ and $\frac{1}{\varrho}$ are not measurable on $\mathbb{R}$ while (1.4) tells us both $\frac{1}{\rho}$ and $\frac{1}{\varrho}$ are measurable on $\mathbb{R}$. So this paper is a continuation of [32]. Furthermore, (1.7) makes both the nonlinearities $t \to p(t)f(t,u,v)$ and $t \to q(t)g(t,u,v)$ be non-Carathéodory functions. An example is presented to show us that the main results in this paper are interesting.

By a solution of BVP(1.5) we mean a couple of functions (x,y) with $x, y \in C^1(t_k, t_{k+1}](k \in \mathbb{Z})$ such that both

$$\Phi(\rho x') : t \to \Phi(\rho(t)x'(t)) \text{ and } \Psi(\varrho y') : t \to \Psi(\varrho(t)y'(t))$$

are derivative on each interval $(t_k, t_{k+1}](k \in \mathbb{Z})$, and the limits

$$\lim_{t\to-\infty} x(t),\ \lim_{t\to-\infty} y(t),\ \lim_{t\to+\infty} \rho(t)x'(t) \text{ and } \lim_{t\to+\infty} \varrho(t)y'(t)$$

exist, and all equations in (1.5) are satisfied. We call (x,y) a positive solution of BVP(1.5) if (x,y) is a solution of BVP(1.5) and $[x(t)]^2 + [y(t)]^2 > 0$ for all $t \in \mathbb{R}$.

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the main results are presented in Section 3. An example is given in Section 4.

2 Preliminary Results

In this section, we present some background definitions. The preliminary results are given too.

Denote

$$\sigma_0(t) = 1 + \int_{-\infty}^{t} \frac{du}{\rho(u)},\ \sigma_1(t) = 1 + \int_{t}^{+\infty} p(s)ds,$$

$$\tau_0(t) = 1 + \int_{-\infty}^{t} \frac{du}{\varrho(u)},\ \tau_1(t) = 1 + \int_{t}^{+\infty} q(s)ds.$$

Definition 2.1. *$h : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a strongly $\varrho-$Carathédory function if it satisfies*

(i) *$t \to h\left(t, \tau_0(t)u, \frac{\Psi^{-1}(\tau_1(t))}{\varrho(t)}v\right)$ is measurable on $\mathbb{R}$ and for each $r > 0$*

$$\lim_{t\to-\infty} h\left(t, \tau_0(t)u, \frac{\Psi^{-1}(\tau_1(t))}{\varrho(t)}v\right) = 0 \text{ uniformly for all } |u|, |v| \le r.$$

(ii) *$(u, v) \to h\left(t, \tau_0(t)u, \frac{\Psi^{-1}(\tau_1(t))}{\varrho(t)}v\right)$ is continuous for a.e. $t \in \mathbb{R}$.*

(iii) *For each $r > 0$, there exists nonnegative number $M_r \ge 0$ such that $|u|, |v| \le r$ implies*

$$\left|h\left(t, \tau_0(t)u, \frac{\Psi^{-1}(\tau_1(t))}{\varrho(t)}v\right)\right| \le M_r, t \in \mathbb{R}.$$

Definition 2.2. *$h : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a strongly $\rho-$Carathédory function if it satisfies*

(i) *$t \to h\left(t, \sigma_0(t)u, \frac{\Phi^{-1}(\sigma_1(t))}{\rho(t)}v\right)$ is measurable on $\mathbb{R}$ and for each $r > 0$*

$$\lim_{t\to-\infty} h\left(t, \sigma_0(t)u, \frac{\Phi^{-1}(\sigma_1(t))}{\rho(t)}v\right) = 0 \text{ uniformly for all } |u|, |v| \le r.$$

(ii) *$(u, v) \to h\left(t, \sigma_0(t)u, \frac{\Phi^{-1}(\sigma_1(t))}{\rho(t)}v\right)$ is continuous for a.e. $t \in \mathbb{R}$.*

(iii) *For each $r > 0$, there exists nonnegative number $M_r \ge 0$ such that $|u|, |v| \le r$ implies*

$$\left|h\left(t, \sigma_0(t)u, \frac{\Phi^{-1}(\sigma_1(t))}{\rho(t)}v\right)\right| \le M_r, t \in \mathbb{R}.$$

Definition 2.3. *$K : \{t_s : s \in \mathbb{Z}\} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a discrete $\varrho-$Carathédory function if it satisfies*

(i) *$(u, v) \to K\left(t_s, \tau_0(t_s)u, \frac{\Psi^{-1}(\tau_1(t_s))}{\varrho(t_s)}v\right)$ is continuous for all $s \in \mathbb{Z}$.*

(ii) *For each $r > 0$, there exists nonnegative constants $N_r \ge 0$ such that $|u|, |v| \le r$ implies*

$$\left|K\left(t_s, \tau_0(t_s)u, \frac{\Psi^{-1}(\tau_1(t_s))}{\varrho(t_s)}v\right)\right| \le N_r \text{ for all } s \in \mathbb{Z}.$$

Definition 2.4. *$H : \{t_s : s \in \mathbb{Z}\} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a discrete ρ-Carathédory function if it satisfies*

(i) *$(u, v) \to H\left(t_s, \sigma_0(t_s)u, \frac{\Phi^{-1}(\sigma_1(t_s))}{\rho(t_s)}v\right)$ is continuous for all $s \in \mathbb{Z}$.*

(ii) *For each $r > 0$, there exists nonnegative constants $N_r \ge 0$ such that $|u|, |v| \le r$ implies*

$$\left|H\left(t_s, \sigma_0(t_s)u, \frac{\Phi^{-1}(\sigma_1(t_s))}{\rho(t_s)}v\right)\right| \le N_r \text{ for all } s \in \mathbb{Z}.$$

Definition 2.5. *[17]. Let E be Banach spaces. An operator $T : E \to E$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

Lemma 2.1. *(i) [Schauder][17]: Let X be a Banach space and $\Omega \subset X$ a nonempty, bounded, open and convex subset of X centered at zero point. Let $T : \overline{\Omega} \to X$ be a completely continuous operator with $T(\overline{\Omega}) \subset \overline{\Omega}$. Then T has a fixed point in $\overline{\Omega}$.*

(ii) [Schaufer][17]: Let X be a Banach space and $T : \overline{\Omega} \to X$ be a completely continuous operator with $\Omega = \{x \in X : x = \lambda Tx \text{ for some } \lambda \in [0,1]\}$ is bounded. Then T has a fixed point in Ω.

Define

$$X = \left\{ x : \begin{array}{c} x_{(t_s,t_{s+1}]} \in C^0(t_s,t_{s+1}], s \in \mathbb{Z}, \\ x'_{(t_s,t_{s+1}]} \in C^0(t_s,t_{s+1}], s \in \mathbb{Z}, \\ \textit{the following limits exist:} \\ \lim\limits_{t \to t_s^+} x(t), \ \lim\limits_{t \to t_s^+} \rho(t)x'(t), s \in \mathbb{Z}, \\ \lim\limits_{t \to \pm\infty} \frac{x(t)}{\sigma_0(t)}, \ \lim\limits_{t \to \pm\infty} \frac{\rho(t)x'(t)}{\Phi^{-1}(\sigma_1(t))} \end{array} \right\}$$

and

$$Y = \left\{ y : \begin{array}{c} y_{(t_s,t_{s+1}]} \in C^0(t_s,t_{s+1}], s \in \mathbb{Z}, \\ y'_{(t_s,t_{s+1}]} \in C^0(t_s,t_{s+1}], s \in \mathbb{Z}, \\ \textit{the following limits exist:} \\ \lim\limits_{t \to t_s^+} y(t), \ \lim\limits_{t \to t_s^+} \varrho(t)y'(t), s \in \mathbb{Z}, \\ \lim\limits_{t \to \pm\infty} \frac{y(t)}{\tau_0(t)}, \ \lim\limits_{t \to \pm\infty} \frac{\varrho(t)y'(t)}{\Psi^{-1}(\tau_1(t))} \end{array} \right\}.$$

For $x \in X$, define $||x|| = ||x||_X = \max\left\{\sup\limits_{t\in\mathbb{R}} \frac{|x(t)|}{\sigma_0(t)}, \ \sup\limits_{t\in\mathbb{R}} \frac{\rho(t)|x'(t)|}{\Phi^{-1}(\sigma_1(t))}\right\}$. *For $y \in Y$, define* $||y|| = ||y||_Y = \max\left\{\sup\limits_{t\in\mathbb{R}} \frac{|y(t)|}{\tau_0(t)}, \ \sup\limits_{t\in\mathbb{R}} \frac{\varrho(t)|y'(t)|}{\Psi^{-1}(\tau_1(t))}\right\}$.

Lemma 2.2. *Suppose that $\int_{-\infty}^t \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)} ds$ is convergent. Then X is a Banach space with the norm $|| \cdot ||_X$ and Y a Banach space with the norm $|| \cdot ||_Y$. $E = X \times Y$ is also a Banach space with the norm $||(x,y)|| = \max\{||x||_X, ||y||_Y\}$ for $(x,y) \in X \times Y$.*

Proof. In fact, it is easy to see that X is a normed linear space. Let $\{x_n\}$ be a Cauchy sequence in X. Then $||x_m - x_n|| \to 0, \ m,n \to +\infty$. It follows that

$$\sup_{t\in\mathbb{R}} \frac{|x_m(t)-x_n(t)|}{\sigma_0(t)} \to 0, m,n \to +\infty,$$

$$\lim_{t\in\mathbb{R}} \frac{\rho(t)|x'_m(t)-x'_n(t)|}{\Phi^{-1}(\sigma_1(t))} \to 0, m,n \to +\infty.$$

So

$$\sup_{t\in(t_s,t_{s+1}]}\frac{|x_m(t)-x_n(t)|}{\sigma_0(t)}\to 0, m,n\to+\infty,\ s\in\mathbb{Z},$$

$$\left|\lim_{t\to\pm\infty}\frac{x_m(t)}{\sigma_0(t)}-\lim_{t\to\pm\infty}\frac{x_n(t)}{\sigma_0(t)}\right|\to 0, m,n\to+\infty,$$

$$\lim_{t\in(t_s,t_{s+1}]}\frac{\rho(t)|x_m'(t)-x_n'(t)|}{\Phi^{-1}(\sigma_1(t))}\to 0, m,n\to+\infty,$$

$$\left|\lim_{t\to\pm\infty}\frac{\rho(t)x_m'(t)}{\Phi^{-1}(\sigma_1(t))}-\lim_{t\to\pm\infty}\frac{\rho(t)x_n'(t)}{\Phi^{-1}(\sigma_1(t))}\right|\to 0, m,n\to+\infty.$$

Then both $\lim_{n\to+\infty}\lim_{t\to\pm\infty}\frac{x_n(t)}{\sigma_0(t)}$ and $\lim_{n\to+\infty}\lim_{t\to\pm\infty}\frac{\rho(t)x_n'(t)}{\Phi^{-1}(\sigma_1(t))}$ exist. Define

$$\overline{x}|_{[t_s,t_{s+1}]}(t)=\begin{cases}\lim_{t\to t_s^+}x(t), t=t_s,\\ x(t),\quad t\in(t_s,t_{s+1}],\end{cases}\qquad \overline{x}'|_{[t_s,t_{s+1}]}(t)=\begin{cases}\lim_{t\to t_s^+}x'(t), t=t_s,\\ x'(t),\quad t\in(t_s,t_{s+1}].\end{cases}$$

We know that $t\to\frac{\overline{x}|_{[t_s,t_{s+1}]}(t)}{\sigma_0(t)}$ is continuous on $[t_s,t_{s+1}](s\in\mathbb{Z})$. Thus $t\to\frac{\overline{x_n}|_{[t_s,t_{s+1}]}(t)}{\sigma_0(t)}$ is a Cauchy sequence in $C[t_s,t_{s+1}]$. Then $\frac{\overline{x_n}|_{[t_s,t_{s+1}]}}{\sigma_0(t)}$ uniformly converges to some $\overline{x_0}_s$ in $C[t_s,t_{s+1}]$ as $n\to+\infty$. Similarly $\frac{\overline{\rho(t)x_n'}|_{[t_s,t_{s+1}]}}{\sigma_1(t)}$ uniformly converges to some $\overline{y_0}_s$ in $C[t_s,t_{s+1}]$ as $n\to+\infty$. Define

$$x_0(t)=\overline{x_0}_s(t),\ y_0(t)=\overline{y_0}_s(t),\ \ t\in(t_s,t_{s+1}],\ s\in\mathbb{Z}.$$

Then x_0,y_0 are defined on $\mathbb{R}$ and is continuous on $(t_s,t_{s+1}]$ and the limits $\lim_{t\to t_s^+}x_0(t)$, $\lim_{t\to t_s^+}y_0(t)(s\in\mathbb{Z})$ exist. Furthermore, we have $\lim_{n\to+\infty}\frac{x_n(t)}{\sigma_0(t)}=x_0(t)$ and $\lim_{n\to+\infty}\frac{\rho(t)x_n'(t)}{\Phi^{-1}(\sigma_1(t))}=y_0(t)$ for every $t\in\mathbb{R}$.

From $\sup_{t\in\mathbb{R}}\frac{|x_m(t)-x_n(t)|}{\sigma_0(t)}\to 0, m,n\to+\infty$, let $m\to+\infty$, we get $\sup_{t\in\mathbb{R}}\left|x_0(t)-\frac{x_n(t)}{\sigma_0(t)}\right|\to 0$ as $n\to+\infty$. So

$$\lim_{t\to\pm\infty}x_0(t)=\lim_{t\to\pm\infty}\lim_{n\pm\infty}\frac{x_n(t)}{\sigma_0(t)}=\lim_{n\to\pm\infty}\lim_{t\to\pm\infty}\frac{x_n(t)}{\sigma_0(t)}$$

exists.

Similarly we have $\sup_{t\in\mathbb{R}}\left|y_0(t)-\frac{\rho(t)x_n'(t)}{\Phi^{-1}(\sigma_1(t))}\right|\to 0$ as $n\to+\infty$. So

$$\lim_{t\to\pm\infty}y_0(t)=\lim_{t\to\pm\infty}\lim_{n\pm\infty}\frac{\rho(t)x_n'(t)}{\Phi^{-1}(\sigma_1(t))}=\lim_{n\to\pm\infty}\lim_{t\to\pm\infty}\frac{\rho(t)x_n'(t)}{\Phi^{-1}(\sigma_1(t))}$$

exists.

Since for some $c_n \in \mathbb{R}$ we have for $t \in (t_s, t_{s+1}]$ that

$$\left|x_n(t) - \sum_{t_s<t} \Delta x_n(t_s) - c_n - \int_{-\infty}^{t} \frac{\Phi^{-1}(\sigma_1(s))y_0(s)}{\rho(s)} ds\right|$$

$$\leq \int_{-\infty}^{t} \left|x_n'(s) - \frac{\Phi^{-1}(\sigma_1(s))y_0(s)}{\rho(s)}\right| ds$$

$$= \int_{-\infty}^{t} \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)} \left|\frac{\rho(s)x_n'(s)}{\Phi^{-1}(\sigma_1(s))} - y_0(s)\right| ds$$

$$\leq \int_{-\infty}^{t} \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)} ds \sup_{t\in\mathbb{R}} \left|\frac{\rho(t)x_n'(t)}{\Phi^{-1}(\sigma_1(t))} - y_0(t)\right|$$

$$\to 0 \text{ as } n \to +\infty.$$

So $\lim\limits_{n\to+\infty}[x_n(t) - \sum\limits_{t_s<t} \Delta x_n(t_s) - c_n] = \int_{-\infty}^{t} \frac{\Phi^{-1}(\sigma_1(s))y_0(s)}{\rho(s)} ds$. Then

$$\lim_{n\to+\infty} \left(\sigma_0(t)x_0(t) - \sum_{t_s<t} \Delta\sigma_0(t_s)x_0(t_s) - c_0\right) = \int_{-\infty}^{t} \frac{\Phi^{-1}(\sigma_1(s))y_0(s)}{\rho(s)} ds.$$

Hence $\frac{\Phi^{-1}(\sigma_1(t))y_0(t)}{\rho(t)} = [\sigma_0(t)x_0(t)]'$ for all $t \in (t_s, t_{s+1}](s \in \mathbb{R})$.

So $t \to \sigma(t)x_0(t)$ is an element in X and $x_n \to x_0$ as $n \to +\infty$. We know that X is a Banach space. Similarly we can prove that Y is a Banach space. So $E = X \times Y$ is a Banach space. $\square$

Lemma 2.3. *Suppose that* $\int_{-\infty}^{t} \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)} ds$ *is convergent. Then* $M \subset X$ *is relatively compact if and only if the following items valid:*

(i) *Both* $\left\{t \to \frac{x(t)}{\sigma_0(t)} : x \in M\right\}$ *and* $\left\{t \to \frac{\rho(t)x'(t)}{\Phi^{-1}(\sigma_1(t))} : x \in M\right\}$ *are uniformly bounded.*

(ii) *Both* $\left\{t \to \frac{x(t)}{\sigma_0(t)} : x \in M\right\}$ *and* $\left\{t \to \frac{\rho(t)x'(t)}{\Phi^{-1}(\sigma_1(t))} : x \in M\right\}$ *are equi-continuous on* $(t_s, t_{s+1}]$ $(s \in \mathbb{Z})$.

(iii) *Both* $\left\{t \to \frac{x(t)}{\sigma_0(t)} : x \in M\right\}$ *and* $\left\{t \to \frac{\rho(t)x'(t)}{\Phi^{-1}(\sigma_1(t))} : x \in M\right\}$ *are equi-convergent as* $t \to \pm\infty$.

Proof. **(⇐)**. From Lemma 2.2, we know X is a Banach space. In order to prove that the subset M is relatively compact in X, we only need to show M is totally bounded in X, that is for all $\epsilon > 0$, M has a finite ϵ-net.

Given $x \in M$, for any given $\epsilon > 0$, by **(i)-(iii)**, there exist constants $A > 0, \delta > 0$, and positive integer s_0 such that

$$\left|\frac{x(u_1)}{\sigma_0(u_1)}-\frac{x(u_2)}{\sigma_0(u_2)}\right|\leq \frac{\epsilon}{3},\ u_1,u_2\leq t_{-s_0} \text{ or } u_1,u_2\geq t_{s_0},$$

$$\left|\frac{\rho(u_1)x'(u_1)}{\Phi^{-1}(\sigma_1(u_1))}-\frac{\rho(u_2)x'(u_2)}{\Phi^{-1}(\sigma_1(u_2))}\right|\leq \frac{\epsilon}{3},\ u_1,u_2\leq t_{-s_0} \text{ or } u_1,u_2\geq t_{s_0},$$

$$||x||=\max\left\{\sup_{t\in\mathbb{R}}\frac{|x(t)|}{\sigma_0(t)},\ \sup_{t\in\mathbb{R}}\frac{\rho(t)x'(t)}{\Phi^{-1}(\sigma_1(t))}\right\}\leq A,$$

$$\left|\frac{x(u_1)}{\sigma_0(u_1)}-\frac{x(u_2)}{\sigma_0(u_2)}\right|\leq \frac{\epsilon}{3},\ u_1,u_2\in(t_s,t_{s+1}],\ |u_1-u_2|<\delta,\ s=-s_0,-s_0+1,\cdots,s_0-1,$$

$$\left|\frac{\rho(u_1)x'(u_1)}{\Phi^{-1}(\sigma_1(u_1))}-\frac{\rho(u_2)x'(u_2)}{\Phi^{-1}(\sigma_1(u_2))}\right|\leq \frac{\epsilon}{3},u_1,u_2\in(t_s,t_{s+1}],|u_1-u_2|<\delta,s=-s_0,-s_0+1,\cdots,s_0-1.$$

Define $X|_{[t_{-s_0},t_{s_0}]}=\Big\{x|_{[t_{-s_0},t_{s_0}]}:x\in X\Big\}$. For $x\in X|_{[t_{-s_0},t_{s_0}]}$, define

$$||x||_{s_0}=\max\left\{\sup_{t\in[t_{-s_0},t_{s_0}]}\frac{|x(t)|}{\sigma_0(t)},\ \sup_{t\in[t_{-s_0},t_{s_0}]}\frac{\rho(t)x'(t)}{\Phi^{-1}(\sigma_1(t))}\right\}.$$

Similarly to Lemma 2.2, we can prove that $X_{[t_{-s_0},t_{s_0}]}$ is a Banach space with the norm $||\cdot||_{s_0}$.

Let $M|_{[t_{-s_0},t_{s_0}]}=\{t\to x(t),t\in[t_{-s_0},t_{s_0}]:x\in M\}$. Then $M|_{[t_{-s_0},t_{s_0}]}$ is a subset of $X|_{[t_{-s_0},t_{s_0}]}$. By Ascoli-Arzela theorem, we can know that $M|_{[t_{-s_0},t_{s_0}]}$ is relatively compact in $X|_{[t_{-s_0},t_{s_0}]}$. Thus, there exist $x_1,x_2,\cdots,x_k\in M$ such that, for any $x\in M$, we have that there exists some $i=1,2,\cdots,k$ such that

$$||x-x_i||_{s_0}=\max\left\{\sup_{t\in[t_{-s_0},t_{s_0}]}\frac{|x(t)-x_i(t)|}{\sigma_0(t)},\ \sup_{t\in[t_{-s_0},t_{s_0}]}\frac{\rho(t)|x'(t)-x_i'(t)|}{\Phi^{-1}(\sigma_1(t))}\right\}\leq\frac{\epsilon}{3}.$$

Therefore, for $x\in M$, we have that

$$||x-x_i||_X=\max\left\{\sup_{t\in\mathbb{R}}\frac{|x(t)-x_i(t)|}{\sigma_0(t)},\ \sup_{t\in\mathbb{R}}\frac{\rho(t)|x'(t)-x_i'(t)|}{\Phi^{-1}(\sigma_1(t))}\right\}$$

$$\leq\max\left\{\sup_{t\leq t_{-s_0}}\frac{|x(t)-x_i(t)|}{\sigma(t)},\ \sup_{t\in[t_{-s_0},t_{s_0}]}\frac{|x(t)-x_i(t)|}{\sigma(t)},\ \sup_{t\geq t_{s_0}}\frac{|x(t)-x_i(t)|}{\sigma(t)}\right.$$

$$\left.\sup_{t\leq t_{-s_0}}\frac{\rho(t)|x'(t)-x_i'(t)|}{\Phi^{-1}(\sigma_1(t))},\ \sup_{t\in[t_{-s_0},t_{s_0}]}\frac{\rho|x'(t)-x_i'(t)|x'(t)}{\Phi^{-1}(\sigma_1(t))},\ \sup_{t\geq t_{s_0}}\frac{\rho(t)|x'(t)-x_i'(t)|}{\Phi^{-1}(\sigma_1(t))}\right\}.$$

We have that

$$\sup_{t\leq t_{-s_0}}\frac{|x(t)-x_i(t)|}{\sigma_0(t)}\leq\sup_{t\leq t_{-s_0}}\left|\frac{x(t)}{\sigma_0(t)}-\frac{x(t_{-s_0})}{\sigma_0(t_{-s_0})}\right|$$

$$+\left|\frac{x(t_{-s_0})}{\sigma_0(t_{-s_0})}-\frac{x_i(t_{-s_0})}{\sigma_0(t_{-s_0})}\right|+\sup_{t\leq t_{-s_0}}\left|\frac{x_i(t_{-s_0})}{\sigma_0(t_{-s_0})}-\frac{x_i(t)}{\sigma_0(t)}\right|<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.$$

Similarly we can prove that

$$\sup_{t\le t_{-s_0}}\frac{\rho(t)x'(t)}{\Phi^{-1}(\sigma_1(t))}<\epsilon,\ \sup_{t\ge t_{s_0}}\frac{|x(t)-x_i(t)|}{\sigma_0(t)}<\epsilon,\ \sup_{t\ge t_{s_0}}\frac{\rho(t)|x'(t)-x_i'(t)|}{\Phi^{-1}(\sigma_1(t))}<\epsilon.$$

Then $||x-x_i||_X<\epsilon$.

So, for any $\epsilon>0$, M has a finite ϵ-net $\{U_{x_1},U_{x_2},\cdots,U_{x_k}\}$, that is, M is totally bounded in X. Hence M is relatively compact in X.

(⇒). Assume that M is relatively compact, then for any $\epsilon>0$, there exists a finite ϵ-net of M. Let the finite ϵ-net be $\{U_{x_1},U_{x_2},\cdots,U_{x_k}\}$ with $x_i\subset M$. Then for any $x\in M$, there exists U_{x_i} such that $x\in U_{x_i}$ and

$$||x||\le||x-x_i||+||x_i||\le\epsilon+\max\{||x_i||:i=1,2,\cdots,k\}.$$

It follows that $\{||x||:x\in M\}$ is uniformly bounded. Then **(i)** holds.

Since the limit $\lim\limits_{t\to t_s^+}\frac{x(t)}{\sigma_0(t)}$ exists, then

$$\frac{\overline{x}(t)}{\sigma_0(t)}=\begin{cases}\lim\limits_{t\to t_s^+}\frac{x(t)}{\sigma_0(t)},t=t_s,\\ \frac{x(t)}{\sigma_0(t)},t\in(t_s,t_{s+1}]\end{cases}\qquad \frac{\rho(t)\overline{x}'(t)}{\Phi^{-1}(\sigma_1(t))}=\begin{cases}\lim\limits_{t\to t_s^+}\frac{\rho(t)x'(t)}{\Phi^{-1}(\sigma_1(t))},t=t_s,\\ \frac{\rho(t)x'(t)}{\Phi^{-1}(\sigma_1(t))},t\in(t_s,t_{s+1}]\end{cases}$$

are continuous on $[t_s,t_{s+1}]$. So for any $\epsilon>0$ there exists $\delta>0$ such that

$$\left|\frac{\overline{x}_i(u_1)}{\sigma_0(u_1)}-\frac{\overline{x}_i(u_2)}{\sigma_0(u_2)}\right|<\epsilon,$$

$$\left|\frac{\rho(u_1)\overline{x}_i'(u_1)}{\Phi^{-1}(\sigma_1(u_1))}-\frac{\rho(u_2)\overline{x}_i'(u_2)}{\Phi^{-1}(\sigma_1(u_2))}\right|<\epsilon,$$

for all $u_1,u_2\in[t_s,t_{s+1}]$ with $|u_1-u_2|<\delta$ and $i=1,2,\cdots,k$. Then for any $\epsilon>0$ there exists $\delta>0$ such that

$$\left|\frac{x_i(u_1)}{\sigma_0(u_1)}-\frac{x_i(u_2)}{\sigma_0(u_2)}\right|<\epsilon,$$

$$\left|\frac{\rho(u_1)x'(u_1)}{\Phi^{-1}(\sigma_1(u_1))}-\frac{\rho(u_2)x'(u_2)}{\Phi^{-1}(\sigma_1(u_2))}\right|<\epsilon,$$

for all $u_1,u_2\in(t_s,t_{s+1}]$ with $|u_1-u_2|<\delta$ and $i=1,2,\cdots,k$.

For $x\in M$, there exists a i such that $x\in U_{x_i}$. Then we have for $u_1,u_2\in(t_s,t_{s+1}]$ $(s\in\mathbb{Z})$ with $|u_1-u_2|<\delta$ that

$$\left|\frac{x(u_1)}{\sigma_0(u_1)}-\frac{x(u_2)}{\sigma_0(u_2)}\right|\le\left|\frac{x(u_1)}{\sigma_0(u_1)}-\frac{x_i(u_1)}{\sigma_0(u_1)}\right|+\left|\frac{x_i(u_1)}{\sigma_0(u_1)}-\frac{x_i(u_2)}{\sigma_0(u_2)}\right|+\left|\frac{x_i(u_2)}{\sigma_0(u_2)}-\frac{x(u_2)}{\sigma_0(u_2)}\right|\le3\epsilon.$$

$\left\{t\to\frac{x(t)}{\sigma_0(t)}:x\in M\right\}$ is equicontinuous in $(t_s,t_{s+1}]$. Similarly we have $\left\{t\to\frac{\rho(t)x'(t)}{\Phi^{-1}(\sigma_1(t))}:x\in M\right\}$ is equicontinuous in $(t_s,t_{s+1}]$. It follows that **(ii)** holds.

Now we prove that **(iii)** holds. It is easily seen that there exists a positive integer s_0 such that

$$\left|\frac{x_i(u_1)}{\sigma_0(u_1)}-\frac{x_i(u_2)}{\sigma_0(u_2)}\right|<\epsilon$$

for all $u_1, u_2 \leq t_{-s_0}$, $i = 1, 2, \cdots, k$. For $x \in M$, there exists a i such that $x \in U_{x_i}$. So

$$\left|\frac{x(u_1)}{\sigma_0(u_1)} - \frac{x(u_2)}{\sigma_0(u_2)}\right| \leq \left|\frac{x(u_1)}{\sigma_0(u_1)} - \frac{x_i(u_1)}{\sigma_0(u_1)}\right| + \left|\frac{x_i(u_1)}{\sigma_0(u_1)} - \frac{x_i(u_2)}{\sigma_0(u_2)}\right|$$

$$+ \left|\frac{x_i(u_2)}{\sigma_0(u_2)} - \frac{x(u_2)}{\sigma_0(u_2)}\right| \leq 3\epsilon,\ u_1, u_2 \leq t_{-s_0}.$$

Then $\left\{t \to \frac{x(t)}{\sigma_0(t)} : x \in M\right\}$ is equi-convergent as $t \to -\infty$, similarly we can prove that is equi-convergent as $t \to +\infty$ and $\left\{t \to \frac{\rho(t)x'(t)}{\Phi^{-1}(\sigma_1(t))} : x \in M\right\}$ are equi-convergent as $t \to \pm\infty$. Hence **(iii)** holds. □

Lemma 2.4. *Suppose that $\int_{-\infty}^{t} \frac{\Psi^{-1}(\tau_1(s))}{\varrho(s)} ds$ is convergent. Then $M \subset Y$ is relatively compact if and only if the following items valid:*

(i) *Both $\left\{t \to \frac{y(t)}{\tau_0(t)} : y \in M\right\}$ and $\left\{t \to \frac{\varrho(t)y'(t)}{\Psi^{-1}(\tau_1(t))} : y \in M\right\}$ are uniformly bounded.*

(ii) *Both $\left\{t \to \frac{y(t)}{\tau_0(t)} : y \in M\right\}$ and $\left\{t \to \frac{\varrho(t)y'(t)}{\Psi^{-1}(\tau_1(t))} : y \in M\right\}$ are equi-continuous on $(t_s, t_{s+1}]$ $(s \in \mathbb{Z})$.*

(iii) *Both $\left\{t \to \frac{y(t)}{\tau_0(t)} : y \in M\right\}$ and $\left\{t \to \frac{\varrho(t)y'(t)}{\Psi^{-1}(\tau_1(t))} : y \in M\right\}$ are equi-convergent as $t \to \pm\infty$.*

Proof. The proof is similar to Lemma 2.3 and is omitted. □

In the sequel, we suppose that $\int_{-\infty}^{t} \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)} ds$ and $\int_{-\infty}^{t} \frac{\Psi^{-1}(\tau_1(s))}{\varrho(s)} ds$ are convergent.

Lemma 2.5. *Suppose that $y \in Y$. Then $u \in X$ is a solution of BVP*

$$\begin{aligned}
&[\Phi(\rho(t)u'(t))]' + p(t)f(t, y(t), y'(t)) = 0, \quad a.e.\ t \in \mathbb{R},\\
&\lim_{t\to-\infty} u(s) = 0,\ \lim_{t\to+\infty} \rho(t)u'(s) = 0,\\
&\Delta u(t_s) = A_{0,s}I_0(t_s, y(t_s), y'(t_s)), \quad s \in \mathbb{Z},\\
&\Delta\Phi(\rho(t_s)u'(t_s)) = A_{1,s}I_1(t_s, y(t_s), y'(t_s)), \quad s \in \mathbb{Z}
\end{aligned} \tag{2.8}$$

if and only if

$$\begin{aligned}
u(t) = &\sum_{t_s<t} A_{0,s}I_0(t_s, y(t_s), y'(t_s))\\
&+ \int_{-\infty}^{t} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty} p(u)f(u, y(u), y'(u))du - \sum_{t_j \geq s} A_{1,j}I_1(t_j, y(t_j), y'(t_j))\right) ds.
\end{aligned} \tag{2.9}$$

Proof. Fix $y \in Y$, then

$$||y|| = \max\left\{\sup_{t\in\mathbb{R}} \frac{|y(t)|}{\tau_0(t)},\ \sup_{t\in\mathbb{R}} \frac{\varrho(t)|y'(t)|}{\Psi^{-1}(\tau_1(t))}\right\} = r < +\infty.$$

Since f is a strongly $\varrho-$Carathéodory function, then there exists nonnegative number $M_r \geq 0$ such that

$$\lim_{t\to-\infty} f\left(t,y(t),y'(t)\right) = \lim_{t\to-\infty} f\left(t,\tau_0(t)\frac{y(t)}{\tau_0(t)},\frac{\Psi^{-1}(\tau_1(t))}{\varrho(t)}\frac{\varrho(t)y'(t)}{\Psi^{-1}(\tau_1(t))}\right)=0,$$
$$|f\left(t,y(t),y'(t)\right)| = \left|f\left(t,\tau_0(t)\frac{y(t)}{\tau_0(t)},\frac{\Psi^{-1}(\tau_1(t))}{\varrho(t)}\frac{\varrho(t)y'(t)}{\Psi^{-1}(\tau_1(t))}\right)\right| \leq M_r, t\in\mathbb{R}. \tag{2.10}$$

Furthermore, $I_0, I_1 : \{t_s : s\in\mathbb{Z}\}\times\mathbb{R}^2\to\mathbb{R}$ are discrete $\varrho-$Carathéodory functions, then there exist nonnegative constants $M_{0,r}, M_{1,r}\geq 0$ such that

$$|I_0\left(t_s,y(t_s),y'(t_s)\right)| \leq M_{0,r}, s\in\mathbb{Z},$$
$$|I_1\left(t_s,y(t_s),y'(t_s)\right)| \leq M_{1,r}, s\in\mathbb{Z}. \tag{2.11}$$

Suppose that $u\in X$ is a solution of (2.8). Then from the boundary conditions we get that

$$\Phi(\rho(t)u'(t)) = \int_t^{+\infty} p(r)f(r,y(r),y'(r))dr - \sum_{t_j\geq t} A_{1,j}I_1(t_j,y(t_j),y'(t_j)),\ t\in\mathbb{R}.$$

So

$$u'(t) = \frac{1}{\rho(t)}\Phi^{-1}\left(\int_t^{+\infty} p(r)f(r,y(r),y'(r))dr - \sum_{t_j>t} A_{1,j}I_1(t_j,y(t_j),y'(t_j))\right).$$

It follows that

$$u(t) = \sum_{t_s<t} A_{0,s}I_0(t_s,y(t_s),y'(t_s))$$
$$+\int_{-\infty}^t \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty} p(u)f(u,y(u),y'(u))du - \sum_{t_j\geq s} A_{1,j}I_1(t_j,x(t_j),y(t_j))\right)ds.$$

So u satisfies (2.9). We now prove that $u\in X$. In fact, we see that

$$u|_{(t_s,t_{s+1}]},\ \ \rho u'|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}],\ s\in\mathbb{Z},$$
$$\lim_{t\to t_s^+} u(t),\ \lim_{t\to t_s^+}\rho(t)u'(t)(s\in\mathbb{Z})\text{ exist.}$$

Now we prove that

$$\lim_{t\to\pm\infty}\frac{u(t)}{\sigma_0(t)}=0,\quad \lim_{t\to\pm\infty}\frac{\rho(t)u'(t)}{\Phi^{-1}(\sigma_1(t))}=0. \tag{2.12}$$

Once sees for $t\in(t_s,t_{s+1}]$ that

$$\frac{|u(t)|}{\sigma_0(t)} \leq \frac{M_{0,r}\sum\limits_{t_s<t}A_{0,s}}{1+\int_{-\infty}^t\frac{du}{\rho(u)}} + \frac{\int_{-\infty}^t\frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty}p(u)M_rdu+\sum\limits_{t_j\geq s}A_{1,j}M_{1,r}\right)ds}{1+\int_{-\infty}^t\frac{du}{\rho(u)}}$$
$$\leq M_{0,r}\frac{\sum\limits_{j=-\infty}^{s}A_{0,j}}{1+\int_{-\infty}^{t_s}\frac{du}{\rho(u)}} + \frac{\int_{-\infty}^t\frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty}p(u)M_rdu+\sum\limits_{t_j\geq s}A_{1,j}M_{1,r}\right)ds}{1+\int_{-\infty}^t\frac{du}{\rho(u)}}.$$

Since

$$\lim_{s\to+\infty}\frac{\sum\limits_{j=-\infty}^{s}A_{0,j}}{1+\int_{-\infty}^{t_s}\frac{du}{\rho(u)}}=\lim_{s\to+\infty}\frac{A_{0,s}}{\int_{t_{s-1}}^{t_s}\frac{du}{\rho(u)}}=0,$$

$$\lim_{t\to+\infty}\frac{\int_{-\infty}^{t}\frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty}p(u)M_r du+\sum\limits_{t_j\geq s}A_{1,j}M_{1,r}\right)ds}{1+\int_{-\infty}^{t}\frac{du}{\rho(u)}}$$

$$=\lim_{t\to+\infty}\Phi^{-1}\left(\int_t^{+\infty}p(u)M_r du+\sum_{t_j\geq t}A_{1,j}M_{1,r}\right)=0,$$

then $\lim\limits_{t\to+\infty}\frac{u(t)}{\sigma_0(t)}=0$. One sees for $t\in[t_{s-1},t_s)$ that

$$\frac{\rho(t)|u'(t)|}{\Phi^{-1}(\sigma_1(t))}\leq\frac{\Phi^{-1}\left(\int_t^{+\infty}p(r)f(r,y(r),y'(r))dr+\sum\limits_{t_s>t}A_{1,s}M_{1,r}\right)}{\Phi^{-1}\left(1+\int_t^{+\infty}p(s)ds\right)}$$

$$=\Phi^{-1}\left(\frac{\int_t^{+\infty}p(r)f(r,y(r),y'(r))dr}{1+\int_t^{+\infty}p(s)ds}+M_{1,r}\frac{\sum\limits_{t_s>t}A_{1,s}}{1+\int_t^{+\infty}p(s)ds}\right)$$

$$\leq\Phi^{-1}\left(\frac{\int_t^{+\infty}p(r)f(r,y(r),y'(r))dr}{1+\int_t^{+\infty}p(s)ds}+M_{1,r}\frac{\sum\limits_{j=s}^{+\infty}A_{1,j}}{1+\int_{t_j}^{+\infty}p(s)ds}\right).$$

Since

$$\lim_{t\to-\infty}\frac{\int_t^{+\infty}p(r)f(r,y(r),y'(r))dr}{1+\int_t^{+\infty}p(s)ds}=\lim_{t\to-\infty}f(t,y(t),y'(t))=0,$$

$$\lim_{s\to-\infty}\frac{\sum\limits_{j=s}^{+\infty}A_{1,j}}{1+\int_{t_j}^{+\infty}p(s)ds}=\lim_{s\to-\infty}\frac{A_{1,s}}{1+\int_{t_s}^{t_{s+1}}p(s)ds}=0,$$

then $\lim\limits_{t\to-\infty}\frac{\rho(t)u'(t)}{\Phi^{-1}(\sigma_1(t))}=0$. It is easy to see from $u(t)$ and $\rho(t)u'(t)$ that $\lim\limits_{t\to-\infty}\frac{u(t)}{\sigma_0(t)}=0$ and $\lim\limits_{t\to+\infty}\frac{\rho(t)u'(t)}{\Phi^{-1}(\sigma_1(t))}=0$. Then $u\in X$.

On the other hand, if u satisfies (2.9), we can prove that $u\in X$ is a solution of BVP(2.8) easily. □

Lemma 2.6. *Suppose that $x \in X$. Then $v \in Y$ is a solution of BVP*

$$\begin{aligned}
&[\Psi(\varrho(t)v'(t))]' + q(t)g(t, x(t), x'(t)) = 0, \quad a.e.\ t \in \mathbb{R},\\
&\lim_{t\to-\infty} v(s) = 0, \quad \lim_{t\to+\infty} \varrho(t)v'(t) = 0,\\
&\Delta v(t_s) = B_{0,s}J_0(t_s, x(t_s), x'(t_s)), \quad s \in \mathbb{Z},\\
&\Delta\Psi(\varrho(t_s)v'(t_s)) = B_{1,s}J_1(t_s, x(t_s), x'(t_s)), \quad s \in \mathbb{Z}
\end{aligned} \tag{2.13}$$

if and only if

$$\begin{aligned}
v(t) &= \sum_{t_s<t} B_{0,s}J_0(t_s, x(t_s), x'(t_s))\\
&+ \int_{-\infty}^{t} \frac{1}{\varrho(s)}\Psi^{-1}\left(\int_s^{+\infty} q(u)g(u, x(u), x'(u))du - \sum_{t_j\geq s} B_{1,j}J_1(t_j, x(t_j), x'(t_j))\right) ds.
\end{aligned} \tag{2.14}$$

Proof. The proof is similar to that of the proof of Lemma 2.5 and is omitted. □

Define the operator T on E by $T(x,y)(t) = (T_1(x,y)(t), T_2(x,y)(t))$ with

$$\begin{aligned}
T_1(x,y)(t) &= \sum_{t_s<t} A_{0,s}I_0(t_s, y(t_s), y'(t_s))\\
&+ \int_{-\infty}^{t} \frac{\Phi^{-1}\left(\int_s^{+\infty} p(u)f(u,y(u),y'(u))du - \sum_{t_j\geq s} A_{1,j}I_1(t_j,y(t_j),y'(t_j))\right)}{\rho(s)} ds,
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
T_2(x,y)(t) &= \sum_{t_s<t} B_{0,s}J_0(t_s, x(t_s), x'(t_s))\\
&+ \int_{-\infty}^{t} \frac{\Psi^{-1}\left(\int_s^{+\infty} q(u)g(u,x(u),x'(u))du - \sum_{t_j\geq s} B_{1,j}J_1(t_j,x(t_j),x'(t_j))\right)}{\varrho(s)} ds.
\end{aligned} \tag{2.16}$$

Lemma 2.7. *The following results hold:*

(i) *$T : E \to E$ is well defined.*

(ii) *$(x,y) \in E$ is a solution of BVP(1.5) if and only if (x,y) is a fixed point of T in E.*

(iii) *$T : E \to E$ is completely continuous.*

Proof. From Lemma 2.2, E is a Banach space. From Lemma 2.5 and Lemma 2.6, we know that (i) and (ii) hold. Note Lemma 2.3 and Lemma 2.4, the proof of (iii) is similar to that of the proof of Lemma 2.4 in [32] and is omitted. □

3 Main Theorems

In this section, the main results on the existence of solutions of BVP(1.5) are established. To establish the first result, we need the following assumption:

(A) There exist nonnegative constants $A_{1,j} \geq 0, B_{1,j} \geq 0 (j = 0, 1, \cdots, m)$, $a_{i,j} \geq 0, b_{i,j} \geq 0 (i = 1, 2, j = 0, 1, \cdots, m)$, and $\tau_{i,j}, \sigma_{i,j} \geq 0 (i = 1, 2, j = 1, 2, \cdots, m)$ satisfying

$$\left|f\left(t, \tau_0(t)u, \frac{\Psi^{-1}(\tau_1(t))}{\varrho(t)}v\right)\right| \leq \Phi\left(A_{1,0} + \sum_{j=1}^{m} A_{1,j}|u|^{\tau_{1,j}}|v|^{\sigma_{1,j}}\right), u, v \in \mathbb{R}, a.e.\ t \in \mathbb{R},$$

$$\left|g\left(t, \sigma_0(t)u, \frac{\Phi^{-1}(\sigma_1(t))}{\rho(t)}v\right)\right| \leq \Psi\left(B_{1,0} + \sum_{j=1}^{m} B_{1,j}|u|^{\tau_{2,j}}|v|^{\sigma_{2,j}}\right), u, v \in \mathbb{R}, a.e.\ t \in \mathbb{R},$$

$$\left|I_0\left(t_s, \tau_0(t_s)u, \frac{\Psi^{-1}(\tau_1(t_s))}{\varrho(t_s)}v\right)\right| \leq a_{1,0} + \sum_{j=1}^{m} a_{1,j}|u|^{\tau_{1,j}}|v|^{\sigma_{1,j}}, u, v \in \mathbb{R}, s \in \mathbb{Z},$$

$$\left|J_0\left(t_s, \sigma_0(t_s)u, \frac{\Phi^{-1}(\sigma_1(t_s))}{\rho(t_s)}v\right)\right| \leq b_{1,0} + \sum_{j=1}^{m} b_{1,j}|u|^{\tau_{2,j}}|v|^{\sigma_{2,j}}, u, v \in \mathbb{R}, s \in \mathbb{Z},$$

$$\left|I_1\left(t_s, \tau_0(t_s)u, \frac{\Psi^{-1}(\tau_1(t_s))}{\varrho(t_s)}v\right)\right| \leq \Phi\left(a_{2,0} + \sum_{j=1}^{m} a_{2,j}|u|^{\tau_{1,j}}|v|^{\sigma_{1,j}}\right), u, v \in \mathbb{R}, s \in \mathbb{Z},$$

$$\left|J_1\left(t_s, \sigma_0(t_s)u, \frac{\Phi^{-1}(\sigma_1(t_s))}{\rho(t_s)}v\right)\right| \leq \Psi\left(b_{2,0} + \sum_{j=1}^{m} b_{2,j}|u|^{\tau_{2,j}}|v|^{\sigma_{2,j}}\right), u, v \in \mathbb{R}, s \in \mathbb{Z}.$$

We denote

$$\sigma = \max\{\tau_{1,j} + \sigma_{1,j},\ \tau_{2,j} + \sigma_{2,j} :\ j = 1, 2, \cdots, m\},$$

$$A_1 = \sup_{s \in \mathbb{Z}} \frac{\sum_{j=-\infty}^{s} A_{0,s}}{1 + \int_{-\infty}^{t_s} \frac{du}{\rho(u)}},$$

$$B_1 = \sup_{t \in \mathbb{R}} \frac{\sigma_{q_1}}{1 + \int_{-\infty}^{t} \frac{du}{\rho(u)}} \int_{-\infty}^{t} \frac{\Phi^{-1}\left(\int_s^{+\infty} p(u)du\right)}{\rho(s)} ds,$$

$$C_1 = \sup_{t \in \mathbb{R}} \frac{\sigma_{q_1}}{1 + \int_{-\infty}^{t} \frac{du}{\rho(u)}} \int_{-\infty}^{t} \frac{\Phi^{-1}\left(\sum_{t_j \geq s} A_{1,j}\right)}{\rho(s)} ds,$$

$$D_1=\sigma_{q_1}\sup_{t\in\mathbb{R}}\Phi^{-1}\left(\frac{\int_t^{+\infty}p(u)du}{1+\int_t^{+\infty}p(s)ds}\right),$$

$$E_1=\sigma_{q_1}\sup_{s\in\mathbb{Z}}\Phi^{-1}\left(\frac{\sum_{j=s+1}^{+\infty}A_{1,j}}{1+\int_{t_{j+1}}^{+\infty}p(s)ds}\right),$$

$$P_1=A_1a_{1,0}+(B_1+D_1)A_{1,0}+(C_1+E_1)a_{2,0},$$

$$Q_i^1=A_1a_{1,i}+(B_1+D_1)A_{1,i}+(C_1+E_1)a_{2,i},i=1,2,\cdots,m$$

and

$$A_2=\sup_{s\in\mathbb{Z}}\frac{\sum_{j=-\infty}^{s}B_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\varrho(u)}},$$

$$B_2=\sup_{t\in\mathbb{R}}\frac{\sigma_{q_2}}{1+\int_{-\infty}^{t}\frac{du}{\varrho(u)}}\int_{-\infty}^{t}\frac{\Psi^{-1}\left(\int_s^{+\infty}q(u)du\right)}{\varrho(s)}ds,$$

$$C_2=\sup_{t\in\mathbb{R}}\frac{\sigma_{q_2}}{1+\int_{-\infty}^{t}\frac{du}{\varrho(u)}}\int_{-\infty}^{t}\frac{\Psi^{-1}\left(\sum_{t_j\geq s}B_{1,j}\right)}{\varrho(s)}ds,$$

$$D_2=\sigma_{q_2}\sup_{t\in\mathbb{R}}\Psi^{-1}\left(\frac{\int_t^{+\infty}q(u)du}{1+\int_t^{+\infty}q(s)ds}\right),$$

$$E_2=\sigma_{q_2}\sup_{s\in\mathbb{Z}}\Psi^{-1}\left(\frac{\sum_{j=s+1}^{+\infty}B_{1,j}}{1+\int_{t_{j+1}}^{+\infty}q(s)ds}\right),$$

$$P_2=A_2a_{2,0}+(B_2+D_2)B_{1,0}+(C_2+E_2)b_{2,0},$$

$$Q_i^2=A_2a_{2,i}+(B_2+D_2)B_{1,i}+(C_2+E_2)b_{2,i},i=1,2,\cdots,m,$$

$$A=\max\{P_1,\ P_2\},$$

$$B=\max\left\{\sum_{i=1}^{m}Q_i^1,\ \sum_{i=1}^{m}Q_i^2\right\}.$$

Theorem 3.1. *Suppose that (a)-(g) and (A) hold and* $\int_{-\infty}^{t}\frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)}ds$ *and* $\int_{-\infty}^{t}\frac{\Psi^{-1}(\tau_1(s))}{\varrho(s)}ds$ *are convergent. Then BVP(1.5) has at least one positive solution if*

(i) $\sigma\in(0,1)$ *or*

(ii) $\sigma=1$ *and* $B<1$ *or*

(iii) $\sigma>1$ *and* $B(A+B)^{\sigma-1}\leq\frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}$.

Proof. We will apply Lemma 2.1 to show the results. Let X,Y,E and T be defined in section 2.

From Lemma 2.7, $T : E \to E$ is a completely continuous operator and $(x, y) \in E$ is a solution of BVP(1.5) if and only if (x, y) is a fixed point of T in E.

For $(x, y) \in E$, we have

$$||(x,y)|| = \max\left\{\sup_{t\in\mathbb{R}} \frac{|x(t)|}{\sigma_0(t)},\ \sup_{t\in\mathbb{R}} \frac{\rho(t)|x'(t)|}{\Phi^{-1}(\sigma_1(t))},\ \sup_{t\in\mathbb{R}} \frac{|y(t)|}{\tau_0(t)},\ \sup_{t\in\mathbb{R}} \frac{\varrho(t)|y'(t)|}{\Psi^{-1}(\tau_1(t))}\right\} = r < +\infty.$$

So

$$|f(t,y(t),y'(t))| = \left|f\left(t, \tau_0(t)\frac{y(t)}{\tau_0(t)}, \frac{\Psi^{-1}(\tau_1(t))}{\varrho(t)}\frac{\varrho(t)y'(t)}{\Psi^{-1}(\tau_1(t))}\right)\right|$$

$$\leq \Phi\left(A_{1,0} + \sum_{j=1}^{m} A_{1,j}\left|\frac{y(t)}{\tau_0(t)}\right|^{\tau_{1,j}}\left|\frac{\varrho(t)y'(t)}{\Psi^{-1}(\tau_1(t))}\right|^{\sigma_{1,j}}\right)$$

$$\leq \Phi\left(A_{1,0} + \sum_{j=1}^{m} A_{1,j}||y||^{\tau_{1,j}+\sigma_{1,j}}\right), a.e.\ t \in \mathbb{R},$$

$$|g\left(t, x(t), x'(t)\right)| \leq \Psi\left(B_{1,0} + \sum_{j=1}^{m} B_{1,j}||x||^{\tau_{2,j}+\sigma_{2,j}}\right), a.e.\ t \in \mathbb{R},$$

$$|I_0\left(t_s, y(t_s), y'(t_s)\right)| \leq a_{1,0} + \sum_{j=1}^{m} a_{1,j}||y||^{\tau_{1,j}+\sigma_{1,j}}, s \in \mathbb{Z},$$

$$|J_0\left(t_s, x(t_s), x'(t_s)\right)| \leq b_{1,0} + \sum_{j=1}^{m} b_{1,j}||x||^{\tau_{2,j}+\sigma_{2,j}}, s \in \mathbb{Z},$$

$$|I_1\left(t_s, y(t_s), y'(t_s)\right)| \leq \Phi\left(a_{2,0} + \sum_{j=1}^{m} a_{2,j}||y||^{\tau_{1,j}+\sigma_{1,j}}\right), s \in \mathbb{Z},$$

$$|J_1\left(t_s, x(t_s), x'(t_s)\right)| \leq \Psi\left(b_{2,0} + \sum_{j=1}^{m} b_{2,j}||x||^{\tau_{2,j}+\sigma_{2,j}}\right), s \in \mathbb{Z}.$$

One knows that

$$\phi_p(u+v) \leq \sigma_p[\phi_p(u) + \phi_p(v)], u, v \geq 0 \text{ with } \sigma_p = \begin{cases} 1,\ 1 < p < 2, \\ 2^{p-1},\ p \geq 2. \end{cases}$$

Then (2.15) implies for $t \in (t_s, t_{s+1}]$ that

$$\frac{|T_1(x,y)(t)|}{\sigma_0(t)} \leq \frac{1}{\sigma_0(t)} \sum_{t_s<t} A_{0,s}|I_0(t_s, y(t_s), y'(t_s))|$$

$$+\frac{1}{\sigma_0(t)} \int_{-\infty}^{t} \frac{1}{\rho(s)}\Phi^{-1}\left(\int_s^{+\infty} p(u)|f(u, y(u), y'(u))|du + \sum_{t_j\geq s} A_{1,j}|I_1(t_j, y(t_j), y'(t_j))|\right) ds$$

$$\leq \frac{\sum_{j=-\infty}^{s} A_{0,s}}{1+\int_{-\infty}^{t_s}\frac{du}{\rho(u)}}\left(a_{1,0}+\sum_{i=1}^{m} a_{1,i}||y||^{\tau_{1,i}+\sigma_{1,i}}\right)$$

$$+\frac{\sigma_q}{1+\int_{-\infty}^{t}\frac{du}{\rho(u)}}\int_{-\infty}^{t}\frac{\Phi^{-1}\left(\int_s^{+\infty}p(u)du\right)}{\rho(s)}ds\left(A_{1,0}+\sum_{i=1}^{m} A_{1,i}||y||^{\tau_{1,i}+\sigma_{1,i}}\right)$$

$$+\frac{\sigma_q}{1+\int_{-\infty}^{t}\frac{du}{\rho(u)}}\int_{-\infty}^{t}\frac{\Phi^{-1}\left(\sum_{t_j\geq s}A_{1,j}\right)}{\rho(s)}ds\left(a_{2,0}+\sum_{i=1}^{m} a_{2,i}||y||^{\tau_{1,i}+\sigma_{1,i}}\right)$$

$$\leq A_1\left(a_{1,0}+\sum_{i=1}^{m} a_{1,i}||(x,y)||^{\tau_{1,i}+\sigma_{1,i}}\right)+B_1\left(A_{1,0}+\sum_{i=1}^{m} A_{1,i}||(x,y)||^{\tau_{1,i}+\sigma_{1,i}}\right)$$

$$+C_1\left(a_{2,0}+\sum_{i=1}^{m} a_{2,i}||(x,y)||^{\tau_{1,i}+\sigma_{1,i}}\right)$$

$$\leq P_1+\sum_{i=1}^{m} Q_i^1||(x,y)||^{\tau_{1,i}+\sigma_{1,i}}.$$

On the other hand, we have for $t\in(t_s,t_{s+1}]$ that

$$\frac{\rho(t)|(T_1(x,y))'(t)|}{\Phi^{-1}(\sigma_1(t))}$$

$$\leq \frac{1}{\Phi^{-1}(\sigma_1(t))}\Phi^{-1}\left(\int_t^{+\infty}p(u)|f(u,y(u),y'(u))|du+\sum_{t_j\geq t}A_{1,j}|I_1(t_j,y(t_j),y'(t_j))|\right)$$

$$\leq \Phi^{-1}\left(\frac{\int_t^{+\infty}p(u)du}{1+\int_t^{+\infty}p(s)ds}\Phi\left(A_{1,0}+\sum_{j=1}^{m}A_{1,j}||y||^{\tau_{1,j}+\sigma_{1,j}}\right)\right.$$

$$\left.+\frac{\sum_{j=s+1}^{+\infty}A_{1,j}}{1+\int_t^{+\infty}p(s)ds}\Phi\left(a_{2,0}+\sum_{j=1}^{m}a_{2,j}||y||^{\tau_{1,j}+\sigma_{1,j}}\right)\right)$$

$$\leq \sigma_q\Phi^{-1}\left(\frac{\int_t^{+\infty}p(u)du}{1+\int_t^{+\infty}p(s)ds}\right)\left(A_{1,0}+\sum_{j=1}^{m}A_{1,j}||y||^{\tau_{1,j}+\sigma_{1,j}}\right)$$

$$+\sigma_q\Phi^{-1}\left(\frac{\sum_{j=s+1}^{+\infty}A_{1,j}}{1+\int_{t_{j+1}}^{+\infty}p(s)ds}\right)\left(a_{2,0}+\sum_{j=1}^{m}a_{2,j}||y||^{\tau_{1,j}+\sigma_{1,j}}\right)$$

$$\leq D_1\left(A_{1,0}+\sum_{i=1}^{m}A_{1,i}||y||^{\tau_{1,i}+\sigma_{1,i}}\right)+E_1\left(a_{2,0}+\sum_{i=1}^{m}a_{2,i}||y||^{\tau_{1,i}+\sigma_{1,i}}\right)$$

$$\leq P_1+\sum_{i=1}^{m} Q_i^1||(x,y)||^{\tau_{1,i}+\sigma_{1,i}}.$$

It follows that

$$\sup_{t\in\mathbb{R}}|T_1(x,y)(t)| \le P_1 + \sum_{i=1}^{m} Q_i^1 ||(x,y)||^{\tau_{1,i}+\sigma_{1,i}}. \tag{3.17}$$

Similarly we get

$$\sup_{t\in\mathbb{R}}|T_2(x,y)(t)| \le P_2 + \sum_{i=1}^{m} Q_i^2 ||(x,y)||^{\tau_{1,i}+\sigma_{1,i}}. \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$||(T_1(x,y), T_2(x,y))|| \le A + B\max\{||(x,y)||^\sigma,\ 1\} \le A + B + B||(x,y)||^\sigma.$$

(i) $\sigma \in (0,1)$.

Since $\sigma \in (0,1)$, change $r_0 > 0$ such that $A + B + Br_0^\sigma \le r_0$. Let $\Omega_0 = \{(x,y) \in X \times Y : ||(x,y)|| \le r_0\}$. Then we get

$$||(T_1(x,y), T_2(x,y))|| \le A + B + Br_0^\sigma \le r_0.$$

So $T\overline{\Omega_0} \subset \overline{\Omega_0}$. Thus Lemma 2.1 implies that the operator T has at least one fixed point in $\overline{\Omega_0}$. So BVP(1.5) has at least one solution.

(ii) $\sigma = 1$ and $B < 1$.

Let $r_0 = \frac{A+B}{1-B}$ such that $A + B + Br_0 = r_0$. Let $\Omega_0 = \{(x,y) \in X \times Y : ||(x,y)|| \le r_0\}$. Then we get

$$||T_1(x,y), T_2(x,y)|| \le A + B + Br_0 \le r_0.$$

So $T\overline{\Omega_0} \subset \overline{\Omega_0}$. Thus Lemma 2.1 implies that the operator T has at least one fixed point in $\overline{\Omega_0}$. So BVP(1.5) has at least one solution.

(iii) $\sigma > 1$ and $B(A+B)^{\sigma-1} \le \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}$.

Let $r_0 = \left(\frac{A+B}{B(\sigma-1)}\right)^{\frac{1}{\sigma}}$. It is easy to show from $\frac{(A+B)^{\sigma-1}\sigma^\sigma}{(\sigma-1)^{\sigma-1}} \le \frac{1}{B}$ that $A + B + Br_0^\sigma \le r_0$. Let $\Omega_0 = \{(x,y) \in X \times Y : ||(x,y)|| \le r_0\}$. Then we get

$$||(T_1(x,y), T_2(x,y))|| \le A + B + Br_0^\sigma \le r_0.$$

So $T\overline{\Omega_0} \subset \overline{\Omega_0}$. Thus Lemma 2.1 implies that the operator T has at least one fixed point (x,y) in $\overline{\Omega_0}$. So BVP(1.5) has at least one solution (x,y).

Since I_0, J_0, f, g are nonnegative functions and I_1, J_1 are non-positive functions, we know by the definition of $T1, T_2$ that (x,y) is a nonnegative solution of BVP(1.5). Furthermore, if there exists $\bar{t} \in \mathbb{R}$ such that $[x(\bar{t})]^2 + [y(\bar{t})]^2 = 0$, suppose that $\bar{t} \in (t_s, t_{s+1}]$ for some $s \in \mathbb{Z}$, from $[\Phi(\rho(t)x'(t))]' = -p(t)f(t, y(t), y'(t)) \le 0$ and $[\Psi(\varrho(t)y'(t))]' = -q(t)g(t, x(t), x'(t)) \le 0$, I_1, J_1 are non-positive, we have $\rho(t)x'(t)$ and $\varrho(t)y'(t)$ is nonincreasing on $\mathbb{R}$. From the boundary conditions in (1.5), we have $\rho(t)x'(t)$ and $\varrho(t)y'(t)$ are nonnegative on $\mathbb{R}$. Since I_0, J_0 are nonnegative and x, y are nondecreasing on $\mathbb{R}$, we have $x(t) \equiv y(t) \equiv 0$ on $(t_s, \bar{t}]$. Then $p(t)f(t,0,0) = q(t)g(t,0,0) = 0$ on $(t_s, \bar{t}]$, a contradiction to (d). Thus (x,y) is a positive solution of BVP(1.5). □

Theorem 3.2. *Suppose that (a)-(g) hold and there exist non-decreasing functions $M_{I_0}, M_{J_0}, M_{I_1}$ and $M_{J_1}, M_f, M_g : \mathbb{R}^3 \to [0,+\infty)$ such that*

$$\left|f\left(t,\tau_0(t)u,\frac{\Psi^{-1}(\tau_1(t))}{\varrho(t)}v\right)\right| \le M_f(|u|,|v|), u,v\in\mathbb{R}, a.e.\ t\in\mathbb{R},$$

$$\left|g\left(t,\sigma_0(t)u,\frac{\Phi^{-1}(\sigma_1(t))}{\rho(t)}v\right)\right| \le M_g(|u|,|v|), u,v\in\mathbb{R}, a.e.\ t\in\mathbb{R},$$

$$\left|I_0\left(t_s,\tau_0(t_s)u,\frac{\Psi^{-1}(\tau_1(t_s))}{\varrho(t_s)}v\right)\right| \le M_{I_0}(|u|,|v|), u,v\in\mathbb{R}, s\in\mathbb{Z},$$

$$\left|J_0\left(t_s,\sigma_0(t_s)u,\frac{\Phi^{-1}(\sigma_1(t_s))}{\rho(t_s)}v\right)\right| \le M_{J_0}(|u|,|v|), u,v\in\mathbb{R}, s\in\mathbb{Z},$$

$$\left|I_1\left(t_s,\tau_0(t_s)u,\frac{\Psi^{-1}(\tau_1(t_s))}{\varrho(t_s)}v\right)\right| \le M_{I_1}(|u|,|v|), u,v\in\mathbb{R}, s\in\mathbb{Z},$$

$$\left|J_1\left(t_s,\sigma_0(t_s)u,\frac{\Phi^{-1}(\sigma_1(t_s))}{\rho(t_s)}v\right)\right| \le M_{J_1}(|u|,|v|), u,v\in\mathbb{R}, s\in\mathbb{Z}.$$

Then BVP(1.5) has at least one positive solution if

$$\begin{aligned}\lim_{r\to+\infty}\frac{1}{r}\Bigg[&M_{J_0}(r,r)\sum_{s=-\infty}^{+\infty}B_{0,s}+\left(1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Psi^{-1}(\tau_1(s))}{\varrho(s)}ds}{\tau_0(t)}\right)\sup_{k\in\mathbb{Z}}\frac{\sum_{s=k+1}^{+\infty}B_{1,s}}{1+\int_{t_{k+1}}^{+\infty}q(s)ds}M_J(r,r)\\&+\left(1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Psi^{-1}(\tau_1(s))}{\varrho(s)}ds}{\tau_0(t)}\right)M_g(r,r)\Bigg]<1,\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\lim_{r\to+\infty}\frac{1}{r}\Bigg[&M_{I_0}(r,r)\sum_{s=-\infty}^{+\infty}A_{0,s}+\left(1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)}ds}{\sigma_0(t)}\right)\sup_{k\in\mathbb{Z}}\frac{\sum_{s=k+1}^{\infty}A_{1,s}}{1+\int_{t_{k+1}}^{+\infty}p(s)ds}M_I(r,r)\\&+\left(1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)}ds}{\sigma_0(t)}\right)M_f(r,r)\Bigg]<1.\end{aligned} \tag{3.20}$$

Proof. Consider the set $\Omega=\{(x,y)\in X\times Y:(x,y)=\lambda T(x,y),\lambda\in[0,1]\}$. For $(x,y)\in\Omega$, we have by the definition of T

$$[\Phi(\rho(t)x'(t))]' + \lambda p(t)f(t,y(t),y'(t)) = 0, \quad a.e.\ t \in \mathbb{R},$$

$$[\Psi(\varrho(t)y'(t))]' + \lambda q(t)g(t,x(t),x'(t)) = 0, \quad a.e.\ t \in \mathbb{R},$$

$$\lim_{t\to-\infty} x(t) = 0, \quad \lim_{t\to+\infty} \rho(t)x'(t) = 0,$$

$$\lim_{t\to-\infty} y(t) = 0, \quad \lim_{t\to+\infty} \varrho(t)y'(t) = 0,$$

$$\Delta x(t_s) = \lambda A_{0,s} I_0(t_s,y(t_s),y'(t_s)),\ \Delta\Phi(\rho(t_s)x'(t_s)) = \lambda A_{1,s} I_1(t_s,y(t_s),y'(t_s)), s \in \mathbb{Z},$$

$$\Delta y(t_s) = \lambda B_{0,s} J_0(t_s,x(t_s),x'(t_s)),\ \Delta\Psi(\varrho(t_s)y'(t_s)) = \lambda B_{1,s} J_1(t_s,x(t_s),x'(t_s)), s \in \mathbb{Z}.$$

From the assumption, we have

$$|f\,(t,y(t),y'(t))| \leq M_f(||y||,||y||), a.e.\ t \in \mathbb{R},$$

$$|g\,(t,x(t),x'(t))| \leq M_g(||x||,||x||), a.e.\ t \in \mathbb{R},$$

$$|I_0\,(t_s,y(t_s),y'(t_s))| \leq M_{I_0}(||y||,||y||), s \in \mathbb{Z},$$

$$|J_0\,(t_s,x(t_s),x'(t_s))| \leq M_{J_0}(||x||,||x||), s \in \mathbb{Z},$$

$$|I_1\,(t_s,y(t_s),y'(t_s))| \leq M_{I_1}(||y||,||y||), s \in \mathbb{Z},$$

$$|J_1\,(t_s,x(t_s),x'(t_s))| \leq M_{J_1}(||x||,||x||), s \in \mathbb{Z}.$$

Then

$$\frac{|x(t)|}{\sigma_0(t)} = \frac{1}{\sigma_0(t)}\left|\int_{-\infty}^t x'(s)ds\right| + \left|\sum_{t_s<t} \lambda A_{0,s} I_0(t_s,y(t_s),y'(t_s))\right|$$

$$\leq \frac{1}{\sigma_0(t)}\int_{-\infty}^t \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)}\frac{\rho(s)|x'(s)|}{\Phi^{-1}(\sigma_1(s))}ds + \sum_{t_s<t} A_{0,s}\,|I_0(t_s,y(t_s),y'(t_s))|$$

$$\leq \frac{1}{\sigma_0(t)}\int_{-\infty}^t \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)}ds \sup_{t\in\mathbb{R}}\frac{\rho(t)|x'(t)|}{\Phi^{-1}(\sigma_1(t))} + \sum_{t_s<t} A_{0,s} M_{I_0}\left(\frac{|y(t_s)|}{\tau_0(t_s)}, \frac{\varrho(t_s)|y'(t_s)|}{\Psi^{-1}(\tau_1(t_s))}\right)$$

$$\leq \frac{1}{\sigma_0(t)}\int_{-\infty}^t \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)}ds \sup_{t\in\mathbb{R}}\frac{\rho(t)|x'(t)|}{\Phi^{-1}(\sigma_1(t))} + \sum_{t_s<t} A_{0,s} M_{I_0}\left(||y||,||y||\right).$$

Since $\lim\limits_{t\to-\infty}\sigma_0(t) = 0$ and $\lim\limits_{t\to+\infty}\sigma_0(t) = +\infty$, we have

$$\lim_{t\to+\infty}\frac{1}{\sigma_0(t)}\int_{-\infty}^t \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)}ds = \lim_{t\to+\infty}\Phi^{-1}(\sigma_1(t)) = 1.$$

Then

$$\frac{|x(t)|}{\sigma_0(t)} \le \sup_{t\in\mathbb{R}} \frac{\int_{-\infty}^{t} \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)} ds}{\sigma_0(t)} \sup_{t\in\mathbb{R}} \frac{\rho(t)|x'(t)|}{\Phi^{-1}(\sigma_1(t))} + M_{I_0}(||y||,||y||) \sum_{s=-\infty}^{+\infty} A_{0,s}.$$

Similarly we have

$$\frac{|y(t)|}{\tau_0(t)} \le \sup_{t\in\mathbb{R}} \frac{\int_{-\infty}^{t} \frac{\Psi^{-1}(\tau_1(s))}{\varrho(s)} ds}{\tau_0(t)} \sup_{t\in\mathbb{R}} \frac{\varrho(t)|y'(t)|}{\Psi^{-1}(\tau_1(t))} + M_{J_0}(||y||,||y||) \sum_{s=-\infty}^{+\infty} A_{1,s}.$$

On the other hand, we have for $t \in (t_k, t_{k+1}]$

$$\frac{|\Phi(\rho(t)x'(t))|}{\sigma_1(t)} = \Phi\left(\frac{\rho(t)|x'(t)|}{\Phi^{-1}(\sigma_1(t))}\right)$$

$$\le \frac{1}{\sigma_1(t)} \sum_{t_s\ge t} |\Delta[\Phi(\rho(t_s)x'(t_s))]| + \frac{1}{\sigma_1(t)} \int_t^{+\infty} |p(s)||f(s,y(s),y'(s))|ds$$

$$\le \frac{1}{\sigma_1(t)} \sum_{t_s\ge t} A_{1,s}|I_1(t_s,y(t_s),y'(t_s))| + \frac{1}{\sigma_1(t)} \int_t^{+\infty} |p(s)||f(s,y(s),y'(s))|ds$$

$$\le \frac{1}{1+\int_{t_{k+1}}^{+\infty} p(s)ds} \sum_{s=k+1}^{+\infty} A_{1,s} M_{I_1}\left(\frac{|y(t_s)|}{\tau_0(t_s)}, \frac{\varrho(t_s)|y'(t_s)|}{\Psi^{-1}(\tau_1(t_s))}\right) + \frac{1}{\sigma_1(t)} \int_t^{+\infty} |p(s)| M_f\left(\frac{|y(s)|}{\tau_0(s)}, \frac{\varrho(s)|y'(s)|}{\Psi^{-1}(\tau_1(s))}\right) ds$$

$$\le \frac{1}{1+\int_{t_{k+1}}^{+\infty} p(s)ds} \sum_{s=k+1}^{+\infty} A_{1,s} M_{I_1}(||y||,||y||) + \frac{1}{\sigma_1(t)} \int_t^{+\infty} |p(s)| M_f(||y||,||y||) ds$$

$$\le \sup_{k\in\mathbb{Z}} \frac{\sum_{s=k+1}^{+\infty} A_{1,s}}{1+\int_{t_{k+1}}^{+\infty} p(s)ds} M_I(||y||,||y||) + M_f(||y||,||y||).$$

It follows that

$$\frac{|x(t)|}{\sigma_0(t)} \le \sup_{t\in\mathbb{R}} \frac{\int_{-\infty}^{t} \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)} ds}{\sigma_0(t)} \sup_{k\in\mathbb{Z}} \frac{\sum_{s=k+1}^{+\infty} A_{1,s}}{1+\int_{t_{k+1}}^{+\infty} p(s)ds} M_I(||y||,||y||) + M_{I_0}(||y||,||y||) \sum_{s=-\infty}^{+\infty} A_{0,s}$$

$$+ \sup_{t\in\mathbb{R}} \frac{\int_{-\infty}^{t} \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)} ds}{\sigma_0(t)} M_f(||y||,||y||).$$

Hence

$$||x|| = \max\left\{\sup_{t\in\mathbb{R}} \frac{|x(t)|}{\sigma_0(t)}, \sup_{t\in\mathbb{R}} \frac{|\Phi(\rho(t)x'(t))|}{\sigma_1(t)}\right\}$$

$$\le M_{I_0}(||y||,||y||) \sum_{s=-\infty}^{+\infty} A_{0,s} + \left[1 + \sup_{t\in\mathbb{R}} \frac{\int_{-\infty}^{t} \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)} ds}{\sigma_0(t)}\right] \sup_{k\in\mathbb{Z}} \frac{\sum_{s=k+1}^{+\infty} A_{1,s}}{1+\int_{t_{k+1}}^{+\infty} p(s)ds} M_I(||y||,||y||)$$

$$+ \left[1 + \sup_{t\in\mathbb{R}} \frac{\int_{-\infty}^{t} \frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)} ds}{\sigma_0(t)}\right] M_f(||y||,||y||).$$

Similarly we have

$$||y|| \leq M_{J_0}(||x||,||x||)\sum_{s=-\infty}^{+\infty} B_{0,s} + \left[1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Psi^{-1}(\tau_1(s))}{\varrho(s)}ds}{\tau_0(t)}\right]\sup_{k\in\mathbb{Z}}\frac{\sum_{s=k+1}^{+\infty}B_{1,s}}{1+\int_{t_{k+1}}^{+\infty}q(s)ds}M_J\left(||x||,||x||\right)$$

$$+\left[1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Psi^{-1}(\tau_1(s))}{\varrho(s)}ds}{\tau_0(t)}\right]M_g\left(||x||,||x||\right).$$

Case 1. If $||x|| \leq ||y||$, then

$$||y|| \leq M_{J_0}(||y||,||y||)\sum_{s=-\infty}^{+\infty} B_{0,s} + \left[1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Psi^{-1}(\tau_1(s))}{\varrho(s)}ds}{\tau_0(t)}\right]\sup_{k\in\mathbb{Z}}\frac{\sum_{s=k+1}^{+\infty}B_{1,s}}{1+\int_{t_{k+1}}^{+\infty}q(s)ds}M_J\left(||y||,||y||\right)$$

$$+\left[1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Psi^{-1}(\tau_1(s))}{\varrho(s)}ds}{\tau_0(t)}\right]M_g\left(||y||,||y||\right).$$

From (3.19), then there exists $M > 0$ such that $||y|| \leq M$. So

$$||x|| \leq M_{I_0}(M,M)\sum_{s=-\infty}^{+\infty} A_{0,s} + \left[1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)}ds}{\sigma_0(t)}\right]\sup_{k\in\mathbb{Z}}\frac{\sum_{s=k+1}^{+\infty}A_{1,s}}{1+\int_{t_{k+1}}^{+\infty}p(s)ds}M_I\left(M,M\right)$$

$$+\left[1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)}ds}{\sigma_0(t)}\right]M_f\left(M,M\right).$$

Then $\Omega = \{(x,y) \in X \times Y : (x,y) = \lambda T(x,y), \lambda \in [0,1]\}$ is bounded. Hence Schaufer's fixed point theorem implies that T has a fixed point (x,y). Similarly to Theorem 3.1, we can prove that (x,y) is a positive solution of BVP(1.5).

Case 2. If $||y|| \leq ||x||$, then

$$||x|| \leq M_{I_0}(||x||,||x||)\sum_{s=-\infty}^{+\infty} A_{0,s} + \left[1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)}ds}{\sigma_0(t)}\right]\sup_{k\in\mathbb{Z}}\frac{\sum_{s=k+1}^{\infty}A_{1,s}}{1+\int_{t_{k+1}}^{+\infty}p(s)ds}M_I\left(||x||,||x||\right)$$

$$+\left[1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Phi^{-1}(\sigma_1(s))}{\rho(s)}ds}{\sigma_0(t)}\right]M_f\left(||x||,||x||\right).$$

From (3.20), then there exists $M > 0$ such that $||x|| \leq M$. So

$$||y|| \leq M_{J_0}(M,M)\sum_{s=-\infty}^{+\infty} B_{0,s} + \left[1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Psi^{-1}(\tau_1(s))}{\varrho(s)}ds}{\tau_0(t)}\right]\sup_{k\in\mathbb{Z}}\frac{\sum_{s=k+1}^{+\infty}B_{1,s}}{1+\int_{t_{k+1}}^{+\infty}q(s)ds}M_J\left(M,M\right)$$

$$+\left[1+\sup_{t\in\mathbb{R}}\frac{\int_{-\infty}^{t}\frac{\Psi^{-1}(\tau_1(s))}{\varrho(s)}ds}{\tau_0(t)}\right]M_g\left(M,M\right).$$

Then $\Omega = \{(x,y) \in X \times Y : (x,y) = \lambda T(x,y), \lambda \in [0,1]\}$ is bounded. Hence Schaufer's fixed point theorem implies that T has a fixed point (x,y). Similarly to Theorem 3.1, we can prove that (x,y) is a positive solution of BVP(1.5). □

4 An example

In this section, we present an example to illustrate the main theorems.

Example 4.1. *Consider the following problem consisting of the differential equations*

$$\begin{aligned} &[(e^{-t}x'(t))^3]' + e^{-2t}\left(A_{1,0} + A_{1,1}\left(\frac{y(t)}{1+\frac{1}{2}e^{2t}}\right)^{\tau_{1,1}}\left(\frac{e^{2t}y'(t)}{\sqrt[5]{1+\frac{1}{2}e^{-2t}}}\right)^{\sigma_{1,1}}\right)^3 = 0, \quad a.e.\ t \in \mathbb{R}, \\ &[(e^{-2t}y'(t))^5]' + e^{-2t}\left(B_{1,0} + B_{1,1}\left(\frac{x(t)}{1+e^{t}}\right)^{\tau_{2,1}}\left(\frac{e^{t}x'(t)}{\sqrt[3]{1+\frac{1}{2}e^{-2t}}}\right)^{\sigma_{2,1}}\right)^5 = 0, \quad a.e.\ t \in \mathbb{R} \end{aligned} \tag{4.21}$$

the boundary conditions

$$\lim_{t\to-\infty} x(t) = 0, \quad \lim_{t\to+\infty} e^t x'(t) = 0, \quad \lim_{t\to-\infty} y(t) = 0, \quad \lim_{t\to+\infty} e^{2t}y'(t) = 0, \tag{4.22}$$

and the impulse effects

$$\begin{aligned} &\Delta x(s) = 2^s\left[a_{1,0} + a_{1,1}\left(\frac{y(s)}{1+\frac{1}{2}e^{2s}}\right)^{\tau_{1,1}}\left(\frac{e^{-2s}y'(s)}{\sqrt[5]{1+\frac{1}{2}e^{-2s}}}\right)^{\sigma_{1,1}}\right], \\ &\Delta(e^{-s}x'(s))^3 = 2^{-s}\left(a_{2,0} + a_{2,1}\left(\frac{y(s)}{1+\frac{1}{2}e^{2s}}\right)^{\tau_{1,1}}\left(\frac{e^{-2s}y'(s)}{\sqrt[5]{1+\frac{1}{2}e^{-2s}}}\right)^{\sigma_{1,1}}\right)^3, s \in \mathbb{Z}, \\ &\Delta y(s) = 3^s\left[b_{1,0} + b_{1,1}\left(\frac{x(s)}{1+e^{s}}\right)^{\tau_{2,1}}\left(\frac{e^{-s}x'(s)}{\sqrt[3]{1+\frac{1}{2}e^{-2s}}}\right)^{\sigma_{2,1}}\right], \\ &\Delta(e^{-2s}y'(s))^5 = 3^{-s}\left(b_{2,0} + b_{2,1}\left(\frac{x(s)}{1+e^{s}}\right)^{\tau_{2,1}}\left(\frac{e^{-s}x'(s)}{\sqrt[3]{1+\frac{1}{2}e^{-2s}}}\right)^{\sigma_{2,1}}\right)^5, s \in \mathbb{Z}. \end{aligned} \tag{4.23}$$

Corresponding to BVP(1.5), we find that

$t_s = s,\ s \in \mathbb{Z},$ *(e) holds,*

$\Phi(x) = x^3,\ \Psi(x) = x^5,\ \Phi^{-1}(x) = x^{\frac{1}{3}},\ \Psi^{-1}(x) = x^{\frac{1}{5}},$

with $p_1 = 3,\ p_2 = 5,\ q_1 = \frac{3}{2},\ q_2 = \frac{5}{4},$ *(c) holds,*

$\rho(t) = e^{-t},\ \varrho(t) = e^{-2t},\ p(t) = e^{-2t},\ q(t) = e^{-2t},$ *(a) and (b) hold,*

$A_{0,s} = 2^s,\ B_{0,s} = 3^s,\ A_{1,s} = 2^{-s},\ B_{1,s} = 3^{-s},\ s \in \mathbb{Z},$ *(g) holds,*

$$\sigma_0(t) = 1 + e^t,\ \sigma_1(t) = 1 + \tfrac{1}{3}e^{-3t},$$

$$\tau_0(t) = 1 + \tfrac{1}{2}e^{2t},\ \tau_1(t) = 1 + \tfrac{1}{2}e^{-2t},$$

$$f(t,u,v) = \left(A_{1,0} + A_{1,1}\left(\frac{|u|}{1+\frac{1}{2}e^{2t}}\right)^{\tau_{1,1}}\left(\frac{e^{2t}|v|}{\sqrt[5]{1+\frac{1}{2}e^{-2t}}}\right)^{\sigma_{1,1}}\right)^3,$$

$$g(t,u,v) = \left(B_{1,0} + B_{1,1}\left(\frac{|u|}{1+e^{t}}\right)^{\tau_{2,1}}\left(\frac{e^{t}|v|}{\sqrt[3]{1+\frac{1}{3}e^{-3t}}}\right)^{\sigma_{2,1}}\right)^5,$$

$$I_0(t_s,u,v) = a_{1,0} + a_{1,1}\left(\frac{|u|}{1+\frac{1}{2}e^{2t_s}}\right)^{\tau_{1,1}}\left(\frac{e^{2t_s}|v|}{\sqrt[5]{1+\frac{1}{2}e^{-2t_s}}}\right)^{\sigma_{1,1}},$$

$$J_0(t_s,u,v) = b_{1,0} + b_{1,1}\left(\frac{|u|}{1+e^{t_s}}\right)^{\tau_{2,1}}\left(\frac{e^{t_s}|v|}{\sqrt[3]{1+\frac{1}{2}e^{-2t_s}}}\right)^{\sigma_{2,1}},$$

$$I_1(t_s,u,v) = -\left(a_{2,0} + a_{2,1}\left(\frac{|u|}{1+\frac{1}{2}e^{2t_s}}\right)^{\tau_{1,1}}\left(\frac{e^{2t_s}|v|}{\sqrt[5]{1+\frac{1}{2}e^{-2t_s}}}\right)^{\sigma_{1,1}}\right)^3,$$

$$J_1(t_s,u,v) = -\left(b_{2,0} + b_{2,1}\left(\frac{|u|}{1+e^{t_s}}\right)^{\tau_{2,1}}\left(\frac{e^{t_s}|v|}{\sqrt[3]{1+\frac{1}{2}e^{-2t_s}}}\right)^{\sigma_{2,1}}\right)^5.$$

One sees that

$$\left|f\left(t,\tau_0(t)u,\frac{\Psi^{-1}(\tau_1(t))}{\varrho(t)}v\right)\right| \le \left(A_{1,0} + A_{1,1}|u|^{\tau_{1,1}}|v|^{\sigma_{1,1}}\right)^3,$$

$$\left|g\left(t,\sigma_0(t)u,\frac{\Phi^{-1}(\sigma_1(t))}{\rho(t)}v\right)\right| \le \left(B_{1,0} + B_{1,1}|u|^{\tau_{2,1}}|v|^{\sigma_{2,1}}\right)^5,$$

$$\left|I_0\left(t_s,\tau_0(t_s)u,\frac{\Psi^{-1}(\tau_1(t_s))}{\varrho(t_s)}v\right)\right| \le a_{1,0} + a_{1,1}|u|^{\tau_{1,1}}|v|^{\sigma_{1,1}},$$

$$\left|J_0\left(t_s,\sigma_0(t_s)u,\frac{\Phi^{-1}(\sigma_1(t_s))}{\rho(t_s)}v\right)\right| \le b_{1,0} + b_{1,1}|u|^{\tau_{2,1}}|v|^{\sigma_{2,1}},$$

$$\left|I_1\left(t_s,\tau_0(t_s)u,\frac{\Psi^{-1}(\tau_1(t_s))}{\varrho(t_s)}v\right)\right| \le \left(a_{2,0} + a_{2,1}|u|^{\tau_{1,1}}|v|^{\sigma_{1,1}}\right)^3,$$

$$\left|J_1\left(t_s,\sigma_0(t_s)u,\frac{\Phi^{-1}(\sigma_1(t_s))}{\rho(t_s)}v\right)\right| \le \left(b_{2,0} + b_{2,1}|u|^{\tau_{2,1}}|v|^{\sigma_{2,1}}\right)^5.$$

One sees that (d), (f) and (A) hold. By direct computation, we get

$$\sigma = \max\{\tau_{1,1} + \sigma_{1,1},\ \tau_{2,1} + \sigma_{2,1}\},$$

$$A_1 = \sup_{s\in\mathbb{Z}} \frac{\sum_{j=-\infty}^{s} 2^j}{1+e^s} = \sup_{s\in\mathbb{Z}} \frac{2^{s+1}}{1+e^s} < 2,$$

$$B_1 = \frac{3}{\sqrt[3]{2}} \sup_{t\in\mathbb{R}} \frac{e^{\frac{1}{3}t}}{1+e^t} < \frac{3}{\sqrt[3]{2}},$$
$$C_1 = \sup_{t\in\mathbb{R}} \frac{1}{1+e^t} \int_{-\infty}^t \frac{\sqrt[3]{\sum_{j=s}^{+\infty} 2^{-j}}}{e^{-s}} ds = \frac{\sqrt[3]{2}}{1-\ln\sqrt[3]{2}} \sup_{t\in\mathbb{R}} \frac{\left(\frac{e}{\sqrt[3]{2}}\right)^t}{1+e^t} < \frac{\sqrt[3]{2}}{1-\ln\sqrt[3]{2}},$$

$$D_1 = \sup_{t\in\mathbb{R}} \sqrt[3]{\frac{\int_t^{+\infty} p(u)du}{1+\int_t^{+\infty} p(s)ds}} < 1,$$

$$E_1 = \sup_{s\in\mathbb{Z}} \sqrt[3]{\frac{\sum_{j=s+1}^{+\infty} 2^{-j}}{1+\int_{s+1}^{+\infty} e^{-2s}ds}} = \sup_{s\in\mathbb{Z}} \sqrt[3]{\frac{2^{-s}}{1+\frac{1}{2}e^{-2(s+1)}}} < 2,$$

$$P_1 < 2a_{1,0} + \left(1+\frac{3}{\sqrt[3]{2}}\right) A_{1,0} + \left(2+\frac{\sqrt[3]{2}}{1-\ln\sqrt[3]{2}}\right) a_{2,0} < 2a_{1,0} + 3.4A_{1,0} + 3.7a_{2,0},$$

$$Q_1^1 < 2a_{1,1} + \left(1+\frac{3}{\sqrt[3]{2}}\right) A_{1,1} + \left(2+\frac{\sqrt[3]{2}}{1-\ln\sqrt[3]{2}}\right) a_{2,1} < 2a_{1,1} + 3.4A_{1,1} + 3.7a_{2,1},$$

and

$$A_2 = \sup_{s\in\mathbb{Z}} \frac{\sum_{j=-\infty}^{s} 3^j}{1+\frac{1}{2}e^{2s}} = \sup_{s\in\mathbb{Z}} \frac{\frac{3}{2}3^s}{1+\frac{1}{2}e^{2s}} < 3,$$

$$B_2 = \frac{5}{8\sqrt[5]{2}} \sup_{t\in\mathbb{R}} \frac{e^{\frac{8}{5}t}}{1+\frac{1}{2}e^{2t}} < \frac{5}{4\sqrt[5]{2}},$$

$$C_2 = \sup_{t\in\mathbb{R}} \frac{1}{1+\frac{1}{2}e^{2t}} \int_{-\infty}^t \frac{\sqrt[5]{\sum_{j=s}^{+\infty} 3^{-j}}}{e^{-2s}} ds = \frac{\sqrt[5]{\frac{3}{2}}}{2-\ln\sqrt[5]{3}} \sup_{t\in\mathbb{R}} \frac{\left(\frac{e^2}{\sqrt[5]{3}}\right)^t}{1+\frac{1}{2}e^{2t}} < \frac{2\sqrt[5]{\frac{3}{2}}}{2-\ln\sqrt[5]{3}},$$

$$D_2 = \sup_{t\in\mathbb{R}} \Psi^{-1}\left(\frac{\int_t^{+\infty} q(u)du}{1+\int_t^{+\infty} q(s)ds}\right) < 1,$$

$$E_2 = \sup_{s\in\mathbb{Z}} \sqrt[5]{\frac{\frac{3}{2}3^{-s-1}}{1+\frac{1}{2}e^{-2(s+1)}}} < 3,$$

$$P_2 < 3a_{2,0} + \left(1+\frac{5}{4\sqrt[5]{2}}\right) B_{1,0} + \left(\frac{2\sqrt[5]{\frac{3}{2}}}{2-\ln\sqrt[5]{3}} + 3\right) b_{2,0} < 3a_{2,0} + 2.1B_{1,0} + 4.3b_{2,0},$$

$$Q_1^2 < 3a_{2,1} + \left(1+\frac{5}{4\sqrt[5]{2}}\right) B_{1,1} + \left(\frac{2\sqrt[5]{\frac{3}{2}}}{2-\ln\sqrt[5]{3}} + 3\right) b_{2,1} < 3a_{2,1} + 2.1B_{1,1} + 4.3b_{2,1},$$

$$A = \max\{P_1,\ P_2\} < 2a_{1,0} + 3.4A_{1,0} + 3.7a_{2,0} + 3a_{2,0} + 2.1B_{1,0} + 4.3b_{2,0},$$

$$B = \max\left\{Q_1^1,\ Q_1^2\right\} < 2a_{1,1} + 3.4A_{1,1} + 3.7a_{2,1} + 3a_{2,1} + 2.1B_{1,1} + 4.3b_{2,1}.$$

By Theorem 3.1, then BVP(4.19)-(4.21) has at least one positive solution if

(i) $\sigma \in (0, 1)$ *or*

(ii) $\sigma = 1$ *and* $2a_{1,1} + 3.4A_{1,1} + 3.7a_{2,1} + 3a_{2,1} + 2.1B_{1,1} + 4.3b_{2,1} < 1$ *or*

(iii) $\sigma > 1$ *and* $2a_{1,1} + 3.4A_{1,1} + 3.7a_{2,1} + 3a_{2,1} + 2.1B_{1,1} + 4.3b_{2,1}(2a_{1,0} + 3.4A_{1,0} + 3.7a_{2,0} + 3a_{2,0} + 2.1B_{1,0} + 4.3b_{2,0} + 2a_{1,1} + 3.4A_{1,1} + 3.7a_{2,1} + 3a_{2,1} + 2.1B_{1,1} + 4.3b_{2,1})^{\sigma-1} \leq \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}$.

Remark 4.1. Similarly to Theorem 3.1 in [32], we can establish existence result of solutions for BVP(1.5) under the assumptions that both $p(t)f(t, u, v)$ and $q(t)g(t, u, v)$ are Carathédory functions. Since both e^{-2t} is not measurable on $\mathbb{R}$, we know that both $t \to e^{-2t}f(t, u, v)$ and $t \to e^{-2t}g(t, u, v)$ are not Carathéodory functions. Then this kind of similar result can not be applied to solve Example 4.1.

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