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A probabilistic proof of an electric network property. Technical Note

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Abstract

We give an elegant probabilistic proof of the fact that multiplying by the same constant the resistance of every resistor in an electric network implies that the effective resistance between any two nodes of the network is multiplied by the same constant.

Key words: Effective resistance; Rayleigh's monotonicity law.

Una prueba probabilística de una propiedad de circuitos eléctricos. Nota Técnica

Resumen

Damos una prueba probabilística elegante de que al multiplicar por una misma constante todas las resistencias de un circuito, la resistencia efectiva entre cualesquiera dos nodos del circuito queda multiplicada por la misma constante.

Palabras clave: Ley de monotonía de Rayleigh; resistencia efectiva.

1. Introduction

On a connected undirected graph $G \quad V, E$ such that the edge between vertices i and j is given a resistance r_{ij} (or equivalently, a conductance $C_{ij} \quad 1/r_{ij}$), we can define the random walk on G as the Markov chain that from its current vertex v jumps to the neighboring vertex with probability $p \quad C \quad / C$, where C v $\quad_{w} C_{vw}$ and $\omega \sim v$ means that ω is a neighbor of v. There may be a conductance C_{zz} from a vertex z to itself, giving rise to a transition probability from z to itself, though the most studied case of these random walks on graphs, the simple random walk, excludes the loops and considers all r_{ij}'s to be equal to 1. Some notation: E_aT_b denotes the expected value, starting from the vertex a, of the hitting time T_b of the vertex b; R_{ab} is the effective resistance, as computed by means of Ohm's law, between vertices a and b.

The beginner's handbook when studying random walks on graphs from the viewpoint of the electric networks is the book of Doyle and Snell (1). Another important reference in this context is the article of Chandra et al. (2) where they prove the following formula for the "commute time" :

$$E_iT_j \quad E_jT_i \quad {}_{z} \; C\, z\, R_{ij}. \qquad [1]$$

2. The theorem

With the facts and notation of the previous section we can prove the following:

Theorem 2.1. Multiplying by the same factor r the resistance of every resistor in an electric network implies that the effective resistance between any two nodes of the network is multiplied by the same factor r.

Remark. The well known Rayleigh's Monotonicity Law (see reference 1) states that if the resistance on any individual resistor of a network is increased (resp. decreased) then the effective resistance between any two nodes of the network is increased (resp. decreased). In this regard, our result can be viewed as a particular case of Rayleigh's Monotonicity Law, with the added fact that we can show the amount by which the effective resistance is increased or decreased.

Proof. We notice that the left hand side of [1] is purely probabilistic, while the right hand side is purely electric. We keep the notation as in the introduction for the original network and denote with a hat the corresponding variables for the transformed network all whose resistors have been multiplied by r. Thus.

$$\hat{p} \quad _{\upsilon\omega} \quad \frac{\hat{C}}{\hat{C}} \quad \frac{\frac{1}{r}C}{\frac{1}{r}C} \quad p \quad .$$

In other words, the probabilistic structure of the transformed network is the same as that of the original network, and thus the left hand side of [1] is the same for the original and the transformed networks.

The right hand side of [1] for the transformed network is

$\frac{1}{r} \; _z \; C \; z \; \hat{R}_{ij}$, and therefore we must have $\hat{R}_{ij} \quad rR_{ij}$, as claimed.

References

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