CIENCIA 8(2), 191-198, 2000
Maracaibo, Venezuela

Unstable properties of difference equations

Licet Lezama and Raúl Naulin*

Departamento de Matemáticas, Universidad de Oriente, Apartado 285. Cumaná, Venezuela.

Recibido: 06-02-99. Aceptado: 18-05-2000

Abstract

In this paper two theorems on instability and asymptotic instability for the null solution of the nonautonomous system of difference equations $x(n+1) = A(n)x(n) + f(n,x(n))$, $f(n,0) = 0$, are proven. The main hypotheses are the existence of an $(1,k)$-dichotomy for the linear system $y(n+1) = A(n)y(n)$ and a monotone condition for $f(n,x)$. The obtained results cover a class of difference systems, which unstable properties cannot be deduced from the classical results on instability of Perron and Coppel.

Key words: Difference equations; discrete dichotomies; instability.

Inestabilidad de ecuaciones en diferencias

Resumen

En este artículo se demuestran dos teoremas de inestabilidad e inestabilidad asintótica de la solución nula del sistema de ecuaciones en diferencias $x(n+1) = A(n)x(n) + f(n,x(n))$, $f(n,0) = 0$. Las hipótesis fundamentales son la existencia de una dicotomía discreta, tipo $(1,k)$, del sistema $y(n+1) = A(n)y(n)$ y una condición de monotonía sobre la función $f(n,x)$. Los resultados obtenidos abarcan una clase de sistemas en diferencias, cuyas propiedades de inestabilidad no pueden ser deducidas de los clásicos teoremas de inestabilidad de Perron y Coppel.

Palabras clave: Ecuaciones en diferencias; dicotomías discretas, inestabilidad.

1. Introduction

Let us consider the nonautonomous system of differente equations

$$y(n+1) = A(n)y(n), \quad n \in N = \{0, 1, 2, 3, \dots\}, \qquad [1]$$

for which all matrices $A(n)$ are invertible. The fundamental matrix Ψ of this system is defined by

$$\Psi(n) = \prod_{s=0}^{n-1} A(s) = A(n-1)\dots A(1)A(0), \prod_{s=0}^{-1} A(s) = I,$$

where I denotes the identity matrix. This paper concerns the unstable properties of the null solution of the nonautonomous difference equation

$$x(n+1) = A(n)x(n) + f(n, x(n)), \quad f(n,0) = 0, \qquad [2]$$

where f is defined on the cylinder $N \times \{x : |x| < H\}$, $H \in (0, \infty]$. This problem has been investigated from the beginning of this century by Perron (1) and Li (2), who established the following

Theorem A (3)

Assume that f(n,x) *is continuous in the variable* x. *Moreover, uniformly with respect to* $n \in N$, *let us assume that*

* Autor para la correspondencia.

$$\lim_{|x|\to 0} \frac{f(n, x)}{|x|} = 0. \quad [3]$$

If A(n) = A = constant and the matrix A has at least one eigenvalue satisfying $|\mu| > 1$, *then the solution x = 0 of Eq. [2] is unstable.*

This result, an important tool in the research of the nonlinear system [2], was improved by Coppel for nonautonomous ordinary differential equations in the article (4), which discrete version can be found in (5):

Theorem B (5)

Assume that f(n,x) is continuous in the variable x and

$$|f(n, y)| \le \gamma |y|, \quad \gamma = constant. \quad [4]$$

Moreover, assume that P, a projection matrix, P $\neq$ *I, satisfies*

$$\sum_{s=n_0}^{n-1} |\Psi(n)P\,\Psi^{-1}(s+1)| + \sum_{s=n}^{\infty} |\Psi(n)(I-P)\Psi^{-1}(s+1)| \le K. \quad [5]$$

where K is a constant. If $K\gamma < 1$, *then the null solution of [2] is unstable.*

Theorem of Coppel yields a criterion of instability for the linear system

$$y(n+1) = [A(n) + B(n)]y(n), \quad n \in N,$$

where the Perron's result turns to be unsuccessful. It is easy to prove that Theorem A follows from Theorem B.

In applications, despite the importance of Theorems A and B, the instability of a large class of system cannot be described by these theorems. The aim of this paper is to provide a method of investigation of the unstable properties of system [2] relying on the dichotomic properties of the nonautonomous system [1]. According to the Coppel's result, this idea seems to be plausible: some king of instability of system [1] must be inherited by system [2], under certain conditions for the coefficient $f(t, x)$. This methodology was proposed in (6) for ordinary differential equations.

In this paper, we will obtain not only the discrete version of the results exposed in (7, 8, 6), but we will communicate two results on the instability of system [2] (Theorem 1 and Theorem 2 of our text) that cannot be obtained from Theorems A and B.

In order to state our basic results, we list our main hypotheses:

(M)

There exists a continuous scalar function $\psi(t, s)$, $t \ge 0$, $s \ge 0$, *monotone nondecreasing in variable s, for each fixed t, such that*

$$|f(n, x)| \le \psi(n, |x|).$$

In what follows $\{k(n)\}$ will denote a sequence of positive numbers.

(D)

System [1] has an $(1, k)$*-dichotomy.*

By this we mean the existence of a projection matrix P such that

$$|\Psi(n)P\,\Psi^{-1}(m)| \le K, \quad 0 \le m \le n$$

$$|\Psi(n)(I-P)\Psi^{-1}(m)| \le Kk(n)k(m)^{-1}, \quad 0 \le n \le m. \quad [6]$$

where K is a constant. In the case $k(n)$=constant, the dichotomy [6] is commonly called an ordinary dichotomy.

The sequence $\{k(n)\}$ stands in the condition **(D)** to characterize an unstable condition of System [1]. We will use the condition

(UNS) $\lim_{n\to\infty} k(n) = \infty.$

The following theorems are the main results of our paper (the notions of Liapounov instability and asymptotic instability are made precise in the next section).

Theorem 1

Let us assume that Eq. [1] has an $(1, k)$*-dichotomy, where k satisfies*

$$k(m) \le Ck(n), \quad n \ge m, \quad C = constant, \qquad [7]$$

and $f(n, x)$ *satisfies* ***(M)****. Moreover, let us assume that there exists a* ρ_0 *such that for* $0 < \rho < \rho_0$ *we have*

$$KC\sum_{s=n_0}^{\infty} \psi(s, k(s)\rho) < \rho. \qquad [8]$$

If $P \ne I$, $\{k(n)^{-1}|\Psi(n)(I-P)|\}$ *is bounded and condition* ***(UNS)*** *is valid, then the null solution of Eq. [2] is unstable.*

Theorem 2

Let us assume that system [1] has an ordinary dichotomy and the nonlinear term $f(n, x)$ *satisfies condition* ***(M)****. Moreover, let us assume that there exists a* ρ_0 *such that for* $0 < \rho < \rho_0$ *we have*

$$K\sum_{s=n_0}^{\infty} \psi(s, \rho) < \rho. \qquad [9]$$

If $P \ne I$, *and* $\{|\Psi(n)(I-P)|\}$ *is bounded, then the null solution of equation [2] is asymptotic unstable.*

These instability results where obtained in (9) for a function $f(t, x)$ satisfying a Lipschitz condition and the linear system [1], possesses an ordinary dichotomy. Under condition (M), the class of system [2] considered here is more general. Our results rely on the application of the Schauder fixed point theorem. Finally, we emphasize that the obtained results are not covered by the cited Theorems A and B and therefore they constitute new results in instability theory.

2. Notations and Preliminaries

For $n_0 \in N$, we will denote $N_{n_0} = \{n \in N: n \ge n_0\}$. In what follows, the sequences $\{y(n, n_0, \xi)\}$ $\{x(n, n_0, \xi)\}$ respectively stand for the solutions of system [1] and [2] with initial condition ξ at the initial time n_0. V denotes the space R^r or C^r with a fixed norm $|\cdot|$. In this paper, the term "sequential space" means a space of sequences which range is contained in V. For a sequence $x: N \to V$, we will denote $|x|_\infty = \sup\{|x(n)|: n \in N\}$ and $|x|_k = |k^{-1}(n)x|_\infty$. The space of all sequences $x: N \to V$ such that $|x|_\infty < \infty$ will be denoted by ℓ^∞, and $\ell_k^\infty = \{x: N \to V, \ |x|_k < \infty\}$. The closed ball in the space ℓ_k^∞, with center in $x = 0$ and radius ρ will be denoted by $B_k[0, \rho] = x \in \ell_k^\infty: |x|_k \le \rho\}$. If $k(n) = constant = 1$ we will abbreviate $B_1[0, \rho] = B[0, \rho]$. In the sequel we will use the following subspaces of initial conditions:

$$V_1 = \{\xi \in V: \ \Psi(n)\xi \in \ell^\infty\},$$

$$V_{1,0} = \{\xi \in V_1: \lim_{n\to\infty} x(n, n_0, \xi) = 0\}.$$

$$V_k = \{\xi \in V: \{k(n)^{-1} x(n, n_0, \xi)\} \in \ell^\infty\}.$$

The notions of stability defined in this paper are the same of (5).

Definition 1

We shall say that the null solution of system [2] *is*

Stable: *If for each* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon, n_0)$, *such that for any,* $|y_0| < \delta$ *implies* $|y(n, n_0, y_0)| < \varepsilon, \forall n \in N_{n_0}$.

Unestable: *If the null Eq.* [2] *solution is not stable.*

Asymtotically stable: *If for any positive* ε *there exists a positive* δ *such that for an initial condition* y_0 *satisfying* $|y_0| < \delta$ *the solution* $y(n, n_0, y_0)$ *is defined on* N_{n_0}, $|y(\cdot, n_0, y_0)|_\infty < \varepsilon$ *and*

$$\lim_{n\to\infty} y(n, n_0, y_0) = 0. \qquad [10]$$

Asymptotically unstable: *If the null solution of Eq.* [2] *is not asymptotically stable.*

Definition 2

Let S be a sequential space endowed with the norm $|\cdot|$. *We will call a set* $\Omega \subset S$ *equiconvergent to* 0, *if only if for every* $\varepsilon > 0$, *there exists* $N \in N$ *such that for all* $x \in S$ *it is satisfied* $|x(n)| < \varepsilon$, *for all* $n \geq N$.

In the following we will use criterion of compactness.

Theorem C

Let S be sequental space. Let Ω *be a bounded, closed and equiconvergent to* 0 *subset of S, then* Ω *is compact.*

In our paper we will use the following version of Schauder fixed point theorem (Theorem 4.4.10 in (10))

Theorem D

Let E be a Banach space with norm $|\cdot|$. *Let* $\mathcal{T}$ *be an operator,* $\mathcal{T}: \Omega \to \Omega$, *where* Ω *is a bounded, closed and convex subset of E. If* $\mathcal{T}(\Omega)$ *is precompact, and* $\mathcal{T}$ *is continuous, then there exists* $x \in \Omega$, *such that* $\mathcal{T}(x) = x$.

The proof of the following result can be found in (11, 12).

Theorem E

Let us assume that system [1] *has an* $(1,k)$*-dichotomy with projection P and the sequence k satisfies the condition* [7], *then this projection can be redefined in order to have the property*

$$\lim_{n\to\infty} k(n)^{-1}\Psi(n)P = 0. \qquad [11]$$

Finally, for futher use, we formally define

$$\mathcal{U}(y)(n) = \sum_{s=n_0}^{n-1} \Psi(n)P\,\Psi^{-1}(s+1)f(s, y(s)) - \sum_{s=n}^{\infty} \Psi(n)(I-P)\Psi^{-1}(s+1)f(s, y(s)).$$

3. Instability

Theorem 3

Let us assume that Eq. [1] *has an* $(1,k)$*-dichotomy, where* $\{k(n)\}$ *satisfies* [7], $f(n,x)$ *satisfies* **(M)** *and* [8] *is valid. If* $V_1 \neq V_k$, *then the null solution of Eq.* [2] *is unstable.*

Proof: Let us assume that the null solution of Eq. [2] is stable. Then for a $\varepsilon > 0$, there exists a $\delta > 0$ such that $|y(n, n_0, y_0)| < \varepsilon$ if $|y_0| < \delta$. Let

$$\rho < \frac{\delta}{k(n_0)}. \qquad [12]$$

Since $V_1 \neq V_k$, then for a small positive σ satisfying

$$\sigma + KC\sum_{s=n_0}^{\infty} \psi(s, k(s)\rho) \leq \rho,$$

we may fix an initial condition $x_0 \in \Psi(n_0)[V_k] \setminus \Psi(n_0)[V_1]$ with the property $|x(n, n_0, x_0)|_k \leq \sigma$. Let us consider the equation

$$y = \mathcal{V}(y),$$

where

$$\mathcal{V}(y)(n) = x(n, n_0, x_0) + \mathcal{U}(y)(n).$$

We will verify the conditions of Theorem D in the space l_k^∞.

S1: The property $\mathcal{V}: B_k[0,\rho] \to B_k[0,\rho]$ follows from [6], [8] and the estimates

$$\begin{aligned} |k(n)^{-1}\mathcal{V}(y)(n)| &\leq |k(n)^{-1}x(n, n_0, x_0)| + k(n)^{-1}|\mathcal{U}(y)(n)| \\ &\leq |k(n)^{-1}x(n, n_0, x_0)| + KC\sum_{s=n_0}^{\infty}\psi(s, k(s)\rho) \\ &\leq \sigma + KC\sum_{s=n_0}^{\infty}\psi(s, k(s)\rho) \leq \rho. \end{aligned}$$

S2: The operator $\mathcal{U}$ is continuous: if $\{y_m\}$ is a sequence contained in ball $B_k[0,\rho]$ converging on N to y_0 in the norm $|x|_k$, then the sequence $\{\mathcal{V}(y_m)\}$ converges to $\{\mathcal{V}(y_0)\}$ in the norm $|x|_k$. For a given $\tilde{\varepsilon} > 0$, we choose a large N such that

$$KC\,k(0)^{-1}\sum_{s=N}^{\infty}\psi(s,k(s)\rho) \le \frac{\tilde{\varepsilon}}{4}$$

Therefore for all $m \in N$ and $n \ge N$, the condition **(M)** implies

$$\left|k(n)^{-1}\sum_{s=N}^{\infty}\Psi(n)(I-P)\Psi^{-1}(s+1)[f(s,y_m(s))-f(s,y(s)]\right|$$

$$\le K\sum_{s=N}^{\infty}k(s)^{-1}(|f(s,y_m(s))|+|f(s,y(s)|)$$

$$\le 2C\;Kk(0)^{-1}\sum_{s=N}^{\infty}\phi(s,k(s)\rho) \le \frac{1}{2}\tilde{\varepsilon}.$$

In a similar manner we obtain

$$\left|k(n)^{-1}\sum_{s=N}^{n-1}|\Psi(n)P\;\Psi^{-1}(s+1)[f(s,y_m(s))-f(s,y(s))]\right| \le \frac{1}{2}\tilde{\varepsilon}.$$

From this last estimates we obtain

$$|k(n)^{-1}[\mathcal{V}(y_m)(n)-\mathcal{V}(y)(n)]| \le$$

$$\le \left|k(n)^{-1}\sum_{s=n_0}^{N-1}\Psi(n)P\;\Psi^{-1}(s+1)[f(s,y_m(s))-f(s,y(s))]\right|$$

$$+\left|k(n)^{-1}\sum_{s=N}^{n-1}|\Psi(n)P\;\Psi^{-1}(s+1)[f(s,y_m(s))-f(s,y(s))]\right|$$

$$+\left|k(n)^{-1}\sum_{s=n-1}^{N}|\Psi(n)(I-P)\Psi^{-1}(s+1)[f(s,y_m(s))-f(s,y(s))]\right|$$

$$+\left|k(n)^{-1}\sum_{s=N}^{\infty}|\Psi(n)(I-P)\Psi^{-1}(s+1)[f(s,y_m(s))-f(s,y(s))]\right|$$

$$\le \sum_{s=n_0}^{N-1}Ck(0)^{-1}\left|\Psi(n)P\;\Psi^{-1}(s+1)[f(s,y_m(s))-f(s,y(s))]\right|_k$$

$$+\sum_{s=n_0}^{N}C\left|\Psi(n)(\mathrm{I}-P)\Psi^{-1}(s+1)[f(s,y_m(s))-f(s,y(s))]\right|_k + \varepsilon.$$

This last expression implies the convergence of $\{\mathcal{V}(y_m)\}$ to $\mathcal{V}(y_0)$ in l_k^∞.

S3: For each sequence $\{y_m\}$ contained in the ball $B_k[0,\rho]$ the sequence $\{\mathcal{V}(y_m)\}$ has a subsequence which converges in l_k^∞: From

$$|\mathcal{U}(y_m)|_k \le \left|\sum_{s=n_0}^{n-1}\Psi(n)P\;\Psi^{-1}(s+1)f(s,y_m(s))\right|_k$$

$$+\left|\sum_{s=n}^{\infty}\Psi(n)(I-P)\;\Psi^{-1}(s+1)f(s,y_m(s))\right|_k,$$

we consider the series

$$\left|\sum_{s=n}^{\infty}\Psi(n)(I-P)\;\Psi^{-1}(s+1)f(s,y_m(s))\right|_k$$

$$\le K\sum_{s=n}^{\infty}k(s+1)^{-1}\psi(s,k(s)\rho)$$

and $\left|\sum_{s=n}^{\infty}\Psi(n)(I-P)\;\Psi^{-1}(s+1)f(s,y_m(s))\right|_k$

$$\le KCk(n_0)^{-1}\sum_{s=n}^{\infty}\psi(s,k(s)\rho).$$

The last expression tends to zero as $n \to \infty$. Therefore, for a large N we have

$$\left|\sum_{s=n}^{\infty}\Psi(n)(I-P)\Psi^{-1}(s+1)f(s,y_m(s))\right|_k < \varepsilon, \forall m, \forall n \ge N.$$

On the other hand

$$\left|\sum_{s=n_0}^{n-1}\Psi(n)P\;\Psi^{-1}(s+1)f(s,y_m(s))\right|_k$$

$$\le KC(n_0)^{-1}\sum_{s=n_0}^{n-1}\psi(s,k(s)\rho).$$

By the dominated convergence theorem (13) and property [11], we obtain

$$\lim_{n\to\infty}k(n)^{-1}\sum_{s=0}^{n-1}\Psi(n)P\;\Psi^{-1}(s+1)f(s,y_m(s)) = 0.$$

Hence

$|k(n)^{-1}\mathcal{U}(y_m)(n)| \le \varepsilon, \forall m, \forall n \ge N.$

Therefore, from Theorem C, the set $\mathcal{V}(B_k[0,\rho])$ is precompact.

From the steps **S1-S3** the conditions of the Theorem D are stisfied, and therefore the operator $\mathcal{V}$ has a fixed point $y(\cdot)$ in the ball $B_k[0,\rho]$. Since $|k(n_0)^{-1}y_0| < \rho$ for [12] we obtain $|y_0| < \delta$, implying that $y(\cdot)$ is a bounded sequence.

We will prove the boundedness of the sequence $\mathcal{U}(y)$. If $y \in B_k[0,\rho]$ then

$$|\mathcal{U}(y)| \le \left|\sum_{s=n_0}^{n-1} \Psi(n)P\ \Psi^{-1}(s+1)f(s, y(s))\right|$$

$$+\left|\sum_{s=n}^{\infty} \Psi(n)(I-P)\ \Psi^{-1}(s+1)f(s, y(s))\right|$$

$$\le KCk(n_0)^{-1}\sum_{s=n_0}^{n-1} \psi(s, k(s)\rho)$$

$$+KCk(n_0)^{-1}\sum_{s=n}^{\infty} \psi(s, k(s)\rho)$$

$$\le KCk(n_0)^{-1}\sum_{s=n_0}^{\infty} \psi(s, k(s)\rho).$$

The condition (8) implies the boundedness of sequence $\mathcal{U}(y)$. Since

$$y(n) = x(n, n_0, x_0) + \mathcal{U}(y)(n),$$

we obtain that the function $x(\cdot, n_0, x_0) + \mathcal{U}(y)(n)$, must be bounded. But this contradicts the choise of x_0. ■

Proof of Theorem 1: follows from Theorem 3, since the conditions $P \ne I$, $\{k(n)^{-1}|\Psi(n)(I-P)|\}$ is bounded and **(UNS)** imply $V_1 \ne V_k$. ■

4. Asympotic Instability

Theorem 4

Let us assume that system [1] *has an ordinary dichotomy,* $f(n,x)$ *satisfies condition* ***(M)*** *and* [9] *is valid.* If $V_1 \ne V_{1,0}$, *then the null solution of equation* [2] *is asymptotic unstable.*

Proof: Let us assume that the null solution of equation [2] is asymptotically stable. This means that for $\varepsilon = 1$ there exists a positive δ such that $|y_0| < \delta$ implies [10]. Let $0 < \rho < \min\{\rho_0, \delta\}$, and let σ be a small number such that

$$\sigma + KC\sum_{s=n_0}^{\infty} \psi(s,\rho) \le \rho.$$

For an initial condition $x_0 \notin \Psi(n_0)[V_1] \setminus \Psi(n_0)[V_{1,0}]$ with $|x(\cdot, n_0, x_0)| < \sigma$, we consider the operator

$$\mathcal{V}(y)(n) = x(n, n_0, x_0) + \mathcal{U}(y)(n).$$

For any $y \in B_k[0,\rho]$ we have estimate

$$|\mathcal{V}(y)(n)| \le \sigma + K\sum_{s=0}^{\infty} \psi(s,\rho) \le \rho, \qquad [13]$$

implying $\mathcal{V}\colon B_0, \rho] \to B[0,\rho]$. By repeating the arguments given in the proof of Therorem 1, we conclude that this operator satisfies the conditions of **Theorem D,** and therefore it has a fixed point $y(\cdot)$ in the ball $B[0,\rho]$; hence

$$y = x(\cdot, n_0, x_0) + \mathcal{U}(y).$$

From **Theorem E,** we may assume that projection P defining the dichotomy [6] satisfies the condition [11]. Therefore

$$y(n) = x(n, n_0, x_0) + o(1). \qquad [14]$$

From [13] the initial condition $y(n)$ satisfies $|y(0)| \le \rho < \delta$ therefore

$$\lim_{n\to\infty} y(n) = 0$$

Under these circumstances, the identity [14] is a contradiction. ■

Proof of Theorem 2: Follows from Theorem 4, since conditions $P \ne I$ and $\{\Psi(n)(I-P)\}$ imply $V_1 \ne V_{1,0}$. ■

5. Examples

Let us consider the difference equation

$$y(n+1)+ = Ay(n) + f(n, y(n)), \qquad [15]$$

where A is a constant matrix. Let us assume that the eigenvalues of this matrix positive satisfying $|\lambda| = 1$ are Jordan simple, that the Jorden boxes corresponding to these eigenvalues are 1-dimensional. Let us define the sets $\sigma_s(A)$, $\sigma_0(A)$ and $\sigma_u(A)$ of eigenvales of A satisfying $|\lambda| < 1$, $|\lambda| = 1$ and $|\lambda| > 1$ respectively. Futher, we assume the condition

$$|f(n, x(n))| \le \gamma(n)|x|^{\alpha},\ \alpha > 0.$$

- If $\sigma_u(A)$ is not empty, $\alpha > 1$ and the sequence $\{\gamma(n)\}$ is bounded, then the null solution of Eq. [15] is unstable.

  This follows from Perron's theorem.

  Let $\sigma_u(A)$ be non empty. Let

  $$1 < R < \min\{|\lambda| : \lambda \in \sigma_u(A)\}$$

  Then system

  $$x(n+1) = Ax(n)$$

  has an $(1, R^n)$-dichotomy. If $\{\gamma(n)\}$ satisfies

  $$\sum_{s=0}^{\infty} \gamma(n) R^{\alpha n} < \infty,$$

  then the null solution of system [15] is unstable.

  This result follows from Theorem 3.

- If $\sigma_u(A)$ is empty, $\sigma_0(A)$ is not empty and the sequence $\{\gamma(n)\}$ satisfies

  $$\sum_{s=0}^{\infty} \gamma(n) < \infty,$$

  then the null solution of system [15] is asymptotically unstable. This result follows from Theorem 4, since $V_1 \ne V_{1,0}$.

As a second example, let us consider the nonautonomous difference equations

$$x(n+2) + a(n)x(n) = \frac{\gamma(n)}{1 + x^2(n)}, \qquad [16]$$

where $\{a(n)\}$ is a real sequence. In order to characterize some dichotomic properties of nonautonomous equation

$$x(n+2) + a(n)x(n) = 0, \qquad [17]$$

let us define the projection $P = \mathrm{diag}\{1, 0\}$. In (9), it is proven that

$$|\Phi(n) P \Phi^{-1}(m)|$$

is bounded for $n \ge m$ if

$$\frac{1}{|g(2m)|} \prod_{k=m}^{n-1} |g(2k)| \le M,\ \ n \ge m,\ \ M = \text{constant}. \qquad [18]$$

On the other hand, we may write the estimate

$$|\Phi(n)(I - P)\Phi^{-1}(m)| \le Mk(n)k(m)^{-1},\ m \ge n,$$

if

$$\frac{1}{|g(2n+1)|} \prod_{k=m}^{m-1} |g(2k+1)| \ge Mk(n)k(m)^{-1},\ m \ge n,$$

M= constant. (19)

Uunder condition [18] and [19], the Eq. [17] has a $(1, k)$-dichotomy.

If $\{k(n)\}$ satisfies the conditions of Therorem 1, then the null solution of Eq. [16] is not stable provided the condition

$$\sum_{n_0}^{\infty} \frac{\gamma(n)}{1 + k(n)^2} < \infty$$

is satisfied

The second example shows that the obtained results of this paper can be applied to the nonautonomous system [1], which linear component [2] does not satisfy consitions [5] of Theorem B.

Acknowledgements

This research was supported by Comisión de Investigación of Oriente University, under Proyecto CI-5-025-00730/95.

References

1. PERRON O. ***J Reine Angew Math*** 161: 41-64, 1929.
2. LI TA. ***Math Zeitschr Bd*** 32: 99-141, 1930.
3. LA SALLE J.P. ***The Stability and Control of Discrete Progress,*** Springer Verlag, Berlin (Germany), 1990.
4. COPPEL W.A. ***J London Math Soc*** 39: 255-260, 1969.
5. AGARWAL R.P. ***Difference Equation and Inequalities (Theory, Methods and Applications), Pure and Applied Mathematics,*** Marcel Dekker, New York (USA), 1992.
6. NAULIN R. ***Differential Equations and Dynamical Systems*** 6(3): 363-376, 1998.
7. AVIS R. ***Electronic Journal of Differential Equations*** 16: 1-7, 1997.
8. LEZAMA L., NAULIN R. Inestabilidad por primera aproximación, to appear in Revista ***Saber***, Universidad de Oriente, Venezuela, 1997.
9. LEZANA L., NAULIN R. ***Annales Univ Sci Budapest*** 41: 1-11, 1998.
10. LLOYD N.G. ***Degree Theory,*** Cambridge University Press, London (UK), 1978.
11. NAULIN R., PINTO M. ***Applicable Analysis*** 69(3-4): 239-255, 1998.
12. COPPEL W.A. ***Dichotomies in Stability Theory,*** Lecture Notes 629, Springer, Berlin (Germany), 1978.
13. ROYDEN H.L. ***Real Analysis,*** Macmillan Publishing Co, New York (USA), 1968.